

# MATRIX

Matrix :-

A matrix is a rectangular arrangement of numbers written in rows and columns.

Ex: (1)  $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$

$$R_1 = [1 \ 2 \ 3 \ 4]$$

$$C_1 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

$$R_2 = [5 \ 6 \ 7 \ 8]$$

$$C_3 = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$C_4 = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

(2)  $B = \begin{bmatrix} 2 & 1 & 7 & -5 \\ 2 & 1 & 6 & -9 \end{bmatrix}$

$$R_1 = [2 \ 1 \ 7 \ -5]$$

$$C_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$R_2 = [2 \ 1 \ 6 \ -9]$$

$$C_3 = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

$$C_4 = \begin{bmatrix} -5 \\ -9 \end{bmatrix}$$

(3)  $C = [1]$

$$R = [1] \quad , \quad C = [1]$$

(4)  $D = [2 \ 1]$

$$R_1 = [2 \ 1]$$

$$C_1 = [2] \quad , \quad C_2 = [1]$$

Order of Matrix :-

order of matrix =  $n \times m$

where  $n$  = number of rows

$m$  = number of

columns



Ex: (1)  $A = \begin{bmatrix} 3 & 8 & 2 & 7 \\ 5 & 9 & 0 & 2 \end{bmatrix}$

$$R_1 = \begin{bmatrix} 3 & 8 & 2 & 7 \end{bmatrix} \quad C_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \quad C_2 = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$$

$$R_2 = \begin{bmatrix} 5 & 9 & 0 & 2 \end{bmatrix} \quad C_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad C_4 = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

Order of  $A = 2 \times 4$

(2)  $B = \begin{bmatrix} 3 & 5 \\ 9 & 2 \end{bmatrix}$

$$R_1 = \begin{bmatrix} 3 & 5 \end{bmatrix} \quad C_1 = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$$

$$R_2 = \begin{bmatrix} 9 & 2 \end{bmatrix} \quad C_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Order of  $B = 2 \times 2$

No. of elements : —

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Ex :

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$a_{11} = 1$$

$$a_{12} = 2$$

$$a_{13} = 3$$

$$a_{21} = 4$$

$$a_{22} = 5$$

$$a_{23} = 6$$

$$a_{31} = 7$$

$$a_{32} = 8$$

$$a_{33} = 9$$



## Types of matrix $\rightarrow$

① Row matrix  $\rightarrow$  Row matrix has exactly one row.

$$\text{ex : } \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

② Column matrix  $\rightarrow$  Column matrix has exactly one column.

$$\text{ex : } \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

③ Zero/null matrix  $\rightarrow$  Matrix whose all elements are zero

$$\text{ex : } \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

④ Horizontal matrix  $\rightarrow$  A matrix of order  $m \times n$  where  $n > m$ .

$$\text{ex : } \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 9 & 5 & 9 \end{bmatrix} 2 \times 4$$

⑤ Vertical matrix  $\rightarrow$  A matrix of order  $m \times n$  where  $m > n$ .

$$\text{ex : } \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix} 4 \times 2$$

⑥ Singleton matrix  $\rightarrow$  The order of the element when  $m \neq n$  is equal.

$$\text{ex : } [1]$$

⑦ Square matrix  $\rightarrow$  In the order have equal no. of rows & columns.

$$\text{ex : } \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} 2 \times 2$$



⑧ Rectangular matrix  $\rightarrow$  If the no. of rows  $\neq$  the no. of columns are not equal.

ex:  $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2 \times 3}$

⑨ Diagonal matrix  $\rightarrow$  The diagonal elements is only numbers but the other elements have to be zero.

ex:  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}_{3 \times 3}$

⑩ Unit matrix  $\rightarrow$  The diagonals have one & others are zero.

ex:  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

⑪ Scalar matrix  $\rightarrow$  The diagonal is said to be a scalar matrix if its diagonal segments are equal.

ex:  $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

⑫ Trace of a matrix  $\rightarrow$  sum of all the diagonals if the elements Denoted by  $[\text{Tr}(A)]$

ex:  $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$   $\text{Tr}(A) = 1 + 4 = 5$

⑬ Comparable matrix  $\rightarrow$  Two matrix  $A = [a_{ij}]_{m \times n}$  &  $B = [b_{ij}]_{p \times q}$  are said to be comparable if  $m=p$  &  $n=q$  are equal.



# ALGEBRA OF MATRICES →

## (A) Addition of matrices :

Let  $A$  &  $B$  two matrices each of same order then their sum  $A+B$  is a matrix of order  $m \times n$  & is obtained by adding the corresponding element of  $A$  &  $B$ .

If  $A = [a_{ij}]_{m \times n}$  &  $B = [b_{ij}]_{m \times n}$ ,

Then  $A+B = [a_{ij} + b_{ij}]_{m \times n}$ ,  $\forall i, j \in \mathbb{N}$

Ex :- ①  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ ,  $B = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$

$A+B$  &  $B+A$  ?

$\therefore A+B = \begin{bmatrix} 4 & 6 \\ 9 & 12 \end{bmatrix}$  &  $B+A = \begin{bmatrix} 4 & 6 \\ 9 & 12 \end{bmatrix}$

②  $A = \begin{bmatrix} 1 & 2 \\ 6 & 8 \end{bmatrix}_{2 \times 2}$ ,  $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2 \times 3}$

$A+B$  &  $B+A$  ?

$\therefore$  solution is not defined.

Properties of matrix addition :-

① Commutative

$$A+B = B+A$$

② Associative

$$A + (B+C) = (A+B) + C$$

③ Identity

$$A+O = O+A \quad [O \text{ identity element}]$$

④ Inverse

$$A+B = O = B+A$$



### (B) scalar multiplication :

Let  $A = [a_{ij}]_{m \times n}$  be a matrix and  $K$  be any no. called a scalar. Then, the matrix obtained by multiplying ~~by~~ every element of  $A$  by  $K$  is called the scalar multiple of  $A$  by  $K$  and is denoted by  $KA$ .

Thus,

$$KA = [Ka_{ij}]_{m \times n}$$

ex :  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$  and  $K = 3$

$$KA = 3 \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$3A = \begin{bmatrix} 1 \times 3 & 2 \times 3 \\ 3 \times 3 & 4 \times 3 \end{bmatrix}$$

$$3A = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$$

### (C) subtraction of matrices :

Let  $A, B$  be two matrices, each of same order  $m \times n$ . Then their subtraction  $A - B$  is a matrix of order  $m \times n$  and is obtained by subtracting the corresponding elements of  $A$  &  $B$ .

Thus, if  $A = [a_{ij}]_{m \times n}$  &  $B = [b_{ij}]_{m \times n}$

$$A - B = [a_{ij} - b_{ij}]_{m \times n} \quad \forall i, j \in \mathbb{N}$$

ex : if  $A = \begin{bmatrix} 0 & 1 \\ 7 & -3 \end{bmatrix}_{2 \times 2}$  &  ~~$B =$~~   $B = \begin{bmatrix} 3 & 5 \\ 8 & 9 \end{bmatrix}_{2 \times 2}$

Find  $A - B$  &  $B - A$



Sol<sup>n</sup>

$$A - B = \begin{bmatrix} 0-3 & 1-5 \\ 7-8 & -3-9 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ -1 & -12 \end{bmatrix}$$

$$B - A = \begin{bmatrix} 3-0 & 5-1 \\ 8-7 & 9-(-3) \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 12 \end{bmatrix}$$

$$\therefore A - B \neq B - A$$

(D) Multiplication of matrices :-

Let  $A$  be a matrix of order  $m \times n$  &  $B$  be a matrix of order  $p \times q$ , then the matrix multiplication  $AB$  is possible only if  $n = p$  & matrices are said to be comparable for multiplication.

$$\text{Let, } A_{m \times n} = [a_{ij}] \text{ & } B_{n \times p} = [b_{ij}] ,$$

then, order of  $AB$  is  $m \times p$  and  $(AB)_{ij} = \sum_{r=1}^n a_{ir} b_{rj}$

$$\text{Ex : } A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 9 & 2 \end{bmatrix}_{2 \times 3} \quad \& \quad B = \begin{bmatrix} 3 & 4 \\ 8 & 1 \\ 0 & 1 \end{bmatrix}_{3 \times 2}$$

$AB$  ?

$$AB = \begin{bmatrix} 1 \times 3 + 3 \times 8 + (-1 \times 0) & 1 \times 4 + 3 \times 1 + (-1 \times 1) \\ 0 \times 3 + 9 \times 8 + 2 \times 0 & 0 \times 4 + 9 \times 1 + 2 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 + 24 + 0 & 4 + 3 - 1 \\ 0 + 72 + 0 & 0 + 9 + 2 \end{bmatrix}$$

$$= \begin{bmatrix} 27 & 6 \\ 72 & 11 \end{bmatrix}$$



## Properties of matrix multiplication :-

- Matrices multiplication is not commutative i.e.,  $AB \neq BA$

Ex:  $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}_{2 \times 2}$        $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{2 \times 2}$

$$AB = \begin{bmatrix} 1 \times 1 + 1 \times 0 & 1 \times 0 + 1 \times 0 \\ 0 \times 1 + 0 \times 0 & 0 \times 0 + 0 \times 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$\therefore AB \neq BA$

- $AB = 0$ , that either  $A = 0$  or  $B = 0$

Ex:  $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ ,  $B = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 \times (-1) + 1 \times 1 & 1 \times 1 + 1 \times (-1) \\ 2 \times (-1) + 2 \times 1 & 2 \times 1 + 2 \times (-1) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- Matrix multiplication is associative if  $A, B$  &  $C$  are comparable for the product  $AB$  &  $BC$ , then  $(AB)C = A(BC)$
- Distributive law :-

$$A(B+C) = AB+AC$$

$$(A+B)C = AC+BC$$

Note:  $A, B$  &  $C$  are comparable for respective product.



- For a square matrix  $A$ ,  
 $A^n = A \cdot A \cdot \dots \cdot A$  where  $n \in \mathbb{N}$

Transpose of matrix :-

Let  $A = [a_{ij}]$  be any matrix of order  $m \times n$ . Then  $A^T$  or  $A' = [a_{ji}]$  for  $1 \leq i \leq m$  and  $1 \leq j \leq n$  of order  $n \times m$ .

Ex : ①  $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$\therefore A^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

②  $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore B^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Orthogonal matrix :-

A square matrix  $A$  is said to be orthogonal matrix, iff  $AA' = I$ , where  $I$  is an identity matrix.

Note :

1. If  $AA' = I$ , then  $A^{-1} = A^T$
2. If  $A$  &  $B$  are orthogonal and then  $AB$  is also orthogonal.
3. If  $A$  is orthogonal, then  $A^{-1}$  &  $A^T$  are orthogonal.
4. The determinant value of orthogonal matrix is either 1 or -1.

Ex :  $A = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$ , check if it is orthogonal.

Soln

$A = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$ ,  $A^T = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$



$$\text{Now, } AXA^T = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$AXA^T = 4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow AXA^T = 4I$$

$\therefore$  Here  $A$  is not orthogonal.

Symmetric matrix :-

A square matrix  $A = [a_{ij}]$  is said to be symmetric if,  
 $a_{ij} = a_{ji} \forall i \neq j$ . So for a symmetric matrix,

$$A = A^T$$

Ex :

$$A = \begin{bmatrix} A & H & G \\ H & B & F \\ G & F & C \end{bmatrix}$$

$$A^T = \begin{bmatrix} A & H & G \\ H & B & F \\ G & F & C \end{bmatrix}$$

$$A = A^T$$

$\therefore A$  is symmetric matrix.

Skew-symmetric matrix :-

A square matrix  $A = [a_{ij}]$  is said to be skew-symmetric if  
 $a_{ij} = -a_{ji} \forall i \neq j$ . For a skew-symmetric matrix  $A = -A^T$ .



Ex :

$$A = \begin{bmatrix} 0 & H & G \\ -H & 0 & F \\ G & -F & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & -H & -G \\ H & 0 & -F \\ G & F & 0 \end{bmatrix}$$

$$\therefore A = -A^T$$

Singular and Non-singular matrices :-

A square matrix  $A$  is said to be a singular, if  $|A| = 0$  and a square matrix  $A$  is said to be non-singular, if  $|A| \neq 0$ .

Ex :

(i)  $A = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$  is a singular matrix, since

$$\Rightarrow |A| = 2 \times 3 - 2 \times 3 = 6 - 6 = 0$$

(ii)  $A = \begin{bmatrix} 5 & 3 \\ 4 & 5 \end{bmatrix}$  is a non-singular matrix, since

$$\Rightarrow |A| = 25 - 12 = 13 \neq 0$$

INVERSE OF A MATRIX :-

Adjoint of a square matrix  $\rightarrow$

Let  $A = [a_{ij}]$  be some  $n \times n$  matrix. The transpose  $B'$  i.e.,  $B = [A_{ij}]$  where,  $A_{ij}$  denotes the cofactor of the element  $a_{ij}$  in the determinant  $|A|$  is called the adjoint of the matrix  $A$  and is denoted by the symbol  $\text{Adj. } A$ .



Thus, the adjoint of a matrix  $A$  is the transpose of the matrix formed by the cofactors of  $A$  i.e., if

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Adjoint of  $A$  = The transpose of the matrix

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix}$$

Note :

- (i) Sometimes the adjoint of a matrix is also called the 'adjugate' of the matrix and  $|\text{adj } A| = |A|^{n-1}$ .
- (ii)  $A (\text{adj } A) = (\text{adj } A) A = |A| I$ , where 'I' is 'identity' matrix

Rule to write cofactors of an element :

Cross the row and column intersection at the element  $a_{ij}$  and the determinant which is left to be denoted by  $D$ .

Then, Cofactors of  $a_{ij} = \begin{cases} D, & \text{if } i+j = \text{even integer} \\ -D, & \text{if } i+j = \text{odd integer} \end{cases}$

Ex :

$$A = \begin{bmatrix} -2 & 2 & 3 \\ -1 & 5 & 10 \\ 4 & 4 & 2 \end{bmatrix}$$



$$A_{11} = 5 \times 2 - 10 \times 4 = 10 - 40 = -30$$

$$A_{12} = -1 \times 2 - 10 \times 4 = -2 - 40 = -42$$

$$A_{13} = -1 \times 4 - 5 \times 4 = -4 - 20 = -24$$

$$A_{21} = 2 \times 2 - 3 \times 4 = 4 - 12 = -8$$

$$A_{22} = -2 \times 2 - 3 \times 4 = -4 - 12 = -16$$

$$A_{23} = -2 \times 4 - 2 \times 4 = -8 - 8 = -16$$

$$A_{31} = 2 \times 10 - 3 \times 5 = 20 - 15 = 5$$

$$A_{32} = -2 \times 10 - 3 \times (-1) = -20 + 3 = -17$$

$$A_{33} = -2 \times 5 - 2 \times (-1) = -10 + 2 = -8$$

$$C_{ij} = (-1)^{i+j} M$$

$$A_{11} = (-1)^{1+1} \times -30 = -30$$

$$A_{12} = (-1)^{1+2} \times -42 = 42$$

$$A_{13} = (-1)^{1+3} \times -24 = -24$$

$$A_{21} = (-1)^{2+1} \times -8 = 8$$

$$A_{22} = (-1)^{2+2} \times -16 = -16$$

$$A_{23} = (-1)^{2+3} \times -16 = 16$$

$$A_{31} = (-1)^{3+1} \times 5 = 5$$

$$A_{32} = (-1)^{3+2} \times -17 = 17$$

$$A_{33} = (-1)^{3+3} \times -8 = -8$$

$$B = \begin{bmatrix} -30 & 42 & -24 \\ 8 & -16 & 16 \\ 5 & 17 & -8 \end{bmatrix}$$

$$\therefore \text{Adj } A = \begin{bmatrix} -30 & 8 & 5 \\ 42 & -16 & 17 \\ -24 & 16 & -8 \end{bmatrix}$$



## Inverse of a matrix (Reciprocal matrix) :-

A square matrix  $A$  (non-singular) of order  $n$  is said to be invertible, if there exists a square matrix  $B$  of the same order,

$$AB = I_n = BA$$

~~Then~~, Then,  $B$  is called the inverse of  $A$  and is denoted by  $A^{-1}$ .

$$\text{Thus, } A^{-1} = B$$

$$A(\text{adj } A) = |A| I_n$$

$$\Rightarrow A^{-1} A (\text{adj } A) = A^{-1} |A| I_n$$

$$\Rightarrow I_n (\text{adj } A) = A^{-1} |A| I_n$$

$$\Rightarrow A^{-1} = \frac{\text{adj } A}{|A|} ; |A| \neq 0$$

Properties of inverse :-

(i) If  $A$  &  $B$  are invertible matrices of the same order, then

$$(AB)^{-1} = A^{-1} B^{-1}$$

(ii) If  $A$  be an invertible matrix, then  $A^T$  is also invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

(iii) If  $A$  is invertible, then

$$(a) (A^{-1})^{-1} = A$$

$$(b) (A^K)^{-1} = (A^{-1})^K ; K \in \mathbb{N}$$

(iv) If  $A$  is non-singular matrix, then

$$|A^{-1}| = |A|^{-1}$$

(v) Orthogonal matrix  $A$  is always invertible and  $A^{-1} = A^T$ .



Note :

For the products  $AB, BA$  to be both defined and be equal, it is necessary that  $A$  and  $B$  are both square matrices of the same order. Thus non-square matrices cannot possess inverse.

Ex (1) If  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ , find  $A^{-1}$

$$A^{-1} = \frac{1}{|A|} \text{adj } A \quad \text{---} \text{adj } A = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$|A| = 1 \times 4 - 2 \times 3 = 4 - 6 = -2$$

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4/-2 & -2/-2 \\ -3/-2 & 1/-2 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

(2) If  $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}$ , find  $A^{-1}$

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$|A| = 1 \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} - 2 \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} + 3 \begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= 1(4-3) - 2(6-1) + 3(9-2)$$

$$= 1 - 10 + 21$$

$$|A| = 12$$



$$C_{ij} = (-1)^{i+j} M_{ij}$$

$$C_{11} = (-1)^{1+1} \cdot 1 = 1$$

$$C_{12} = (-1)^{1+2} \cdot 5 = -5$$

$$C_{13} = (-1)^{1+3} \cdot 7 = 7$$

$$C_{21} = (-1)^{2+1} \cdot -5 = 5$$

$$C_{22} = (-1)^{2+2} \cdot -1 = -1$$

$$C_{23} = (-1)^{2+3} \cdot 1 = -1$$

$$C_{31} = (-1)^{3+1} \cdot (-4) = -4$$

$$C_{32} = (-1)^{3+2} \cdot (-8) = 8$$

$$C_{33} = (-1)^{3+3} \cdot (-4) = -4$$

$$B = \begin{bmatrix} 1 & -5 & 7 \\ 5 & -1 & -1 \\ -4 & 8 & -4 \end{bmatrix}$$

$$\text{Adj } A = \begin{bmatrix} 1 & 5 & -4 \\ -5 & -1 & 8 \\ 7 & -1 & -4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$= \frac{1}{12} \begin{bmatrix} 1 & 5 & -4 \\ -5 & -1 & 8 \\ 7 & -1 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1/12 & 5/12 & -4/12 \\ -5/12 & -1/12 & 8/12 \\ 7/12 & -1/12 & -4/12 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} 1/12 & 5/12 & -1/3 \\ -5/12 & -1/12 & 2/3 \\ 7/12 & -1/12 & -1/3 \end{bmatrix}$$



## Matrix Method :-

(Use of ~~the~~ inverse of a matrix to find the solution of a system of linear equations)

consider a system of  $n$  linear equation in ' $n$ ' unknowns  $x_1, x_2, \dots, x_n$ .

$$\text{i.e., } a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i$$

These equations can be written in the form of a single matrix equation  $AX = B$ .

$$\text{where, } A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}_{n \times n}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}_{n \times 1}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix}_{n \times 1}$$

Suppose  $A$  is a non-singular matrix

$$\text{i.e., } |A| \neq 0$$

Then  $A^{-1}$  exists and  $X = A^{-1}B$ .

$$\text{Ex : } x + y + z = 6$$

$$x - y + z = 2$$

$$2x + y - z = 1$$

Solve the system by matrix method.

Sol<sup>n</sup>

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$



$$|A| = 1 \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= 1(1-1) - 1(-1-2) + 1(1+2)$$

$$= 0 + 3 + 3$$

$$= 6$$

$$m_{11} = 1 - 1 = 0$$

$$m_{12} = -1 - 2 = -3$$

$$m_{13} = 1 + 2 = 3$$

$$m_{21} = -1 - 1 = -2$$

$$m_{22} = -1 - 2 = -3$$

$$m_{23} = 1 - 2 = -1$$

$$m_{31} = 1 + 1 = 2$$

$$m_{32} = 1 - 1 = 0$$

$$m_{33} = -1 - 1 = -2$$

$$C_{11} = (-1)^{1+1} 0 = 0$$

$$C_{12} = (-1)^{1+2} (-3) = 3$$

$$C_{13} = (-1)^{1+3} (3) = 3$$

$$C_{21} = (-1)^{2+1} (-2) = 2$$

$$C_{22} = (-1)^{2+2} (-3) = -3$$

$$C_{23} = (-1)^{2+3} (-1) = 1$$

$$C_{31} = (-1)^{3+1} (2) = 2$$

$$C_{32} = (-1)^{3+2} (0) = 0$$

$$C_{33} = (-1)^{3+3} (-2) = -2$$

$$A = \begin{bmatrix} 0 & 3 & 3 \\ 2 & -3 & 1 \\ 2 & 0 & -2 \end{bmatrix}$$

$$\text{Adj } A = \begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj } A}{|A|}$$

$$A^{-1} = \frac{1}{6} \begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix}$$



$$A^{-1} = \begin{bmatrix} 0/6 & 2/6 & 2/6 \\ 3/6 & -3/6 & 0/6 \\ 3/6 & 1/6 & -2/6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1/3 & 1/3 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/6 & -1/3 \end{bmatrix}$$

$$X = A^{-1}B$$

$$= \begin{bmatrix} 0 & 1/3 & 1/3 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/6 & -1/3 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} (0 \times 6) + (1/3 \cdot 2) + (1/3 \cdot 1) \\ (1/2 \cdot 6) + (-1/2 \cdot 2) + (0 \cdot 1) \\ (1/2 \cdot 6) + (1/6 \cdot 2) + (-1/3 \cdot 1) \end{bmatrix}$$

$$= \begin{bmatrix} 0 + \frac{2}{3} + \frac{1}{3} \\ 3 - 1 + 0 \\ 3 + \frac{1}{3} - \frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 3/3 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\therefore x = 1, y = 2 \text{ \& } z = 3$$



## DETERMINANT

Determinants are depended as a scalar values associated with square matrices, to every square matrix  $A = [a_{ij}]$  of order  $n$ , you can associate a number (real/complex) called determinant of the square matrix  $A$ .

→ It is denoted by  $\det A$  or  $|A|$  or  $\Delta$ .

$$\text{Ex : (i) } A = \begin{vmatrix} 4 & 3 \\ 7 & 8 \end{vmatrix}$$

$$\begin{aligned} |A| &= 32 - 21 \\ &= 11 \end{aligned}$$

$$\text{(ii) } A = \begin{vmatrix} 4i & 3i \\ 7i & 8i \end{vmatrix}$$

$$\begin{aligned} |A| &= 32(i)^2 - 21(i)^2 \\ &= 32(-1) - 21(-1) \\ &= -32 + 21 \\ &= -11 \end{aligned}$$

$$\boxed{i^2 = -1}$$

Determinant of order  $-1$  :-

Those are trivial determinants and the value of determinant is just the number in the matrix they have 1 row & 1 column.

$$\text{For ex : } A = [7]$$

$$\therefore |A| = 7$$



Determinant of order-2 :-

Consider the system of eq<sup>n</sup>  $a_1x + b_1 = 0$  &  $a_2x + b_2 = 0$ .

→ If those are satisfied by the same value of  $x$ , then

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$= a_1b_2 - b_1a_2$  (value of determinant of order-2)

Ex : (I)  $A = \begin{vmatrix} 2 & 4 \\ 1 & 9 \end{vmatrix}$

$$\Rightarrow 18 - 4 = 14$$

(II)  $A = \begin{vmatrix} 2 & 10 \\ 0 & 7 \end{vmatrix}$

$$\Rightarrow 14 - 0 = 14$$

Note :

Sign system for determinant of order-2 is  $\begin{vmatrix} + & - \\ - & + \end{vmatrix}$ .

Ex :  $A = \begin{vmatrix} x & 3 \\ 4 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix}$

$$\Rightarrow 2x - 12 = 6 - 2$$

$$\Rightarrow 2x = 4 + 12$$

$$\Rightarrow 2x = 16$$

$$\Rightarrow x = 16/2$$

$$\Rightarrow x = 8$$



Determinant of order - 3 :-

Consider the system of eq<sup>n</sup>  $a_1x + b_1y + c_1 = 0$ ,  $a_2x + b_2y + c_2 = 0$   
&  $a_3x + b_3y + c_3 = 0$

→ If these eq<sup>n</sup>s are satisfied by the same value of  $x$  &  $y$ , then

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$= a_1(b_2c_3 - c_2b_3) - b_1(a_2c_3 - c_2a_3) + c_1(a_2b_3 - b_2a_3)$  is called the value of determinant of order - 3.

Note-1:

sign system for det of order 3 :  $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

Note-2:

Modulus function is totally different than determinants.

For ex :  $| -2 | = 2$  in modulus function

$| -2 | = -2$  in determinant

Ex :

$$A = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 3 & 3 \\ 2 & -1 & 1 \end{vmatrix}$$

$$= 1(3+3) - 2(-1-6) + 3(1-6)$$

$$= 6 + 14 - 15$$

$$= 20 - 15$$

$$= 5$$



Value / Expansion of determinants of  $3 \times 3$  order :-

(a) with respect to first row,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - c_2b_3) - b_1(a_2c_3 - c_2a_3) + c_1(a_2b_3 - b_2a_3)$$

Ex: Evaluate  $\begin{vmatrix} 1 & 2 & 3 \\ -1 & 3 & 3 \\ 2 & -1 & 1 \end{vmatrix}$

$$= 1(3 \times 1 - 3 \times (-1)) - 2((-1) \times 1 - 3 \times 2) + 3((-1) \times (-1) - 3 \times 2)$$

$$= 1(3 + 3) - 2(-1 - 6) + 3(1 - 6)$$

$$= 6 + 14 - 15$$

$$= 20 - 15$$

$$= 5$$

(b) with respect to second column

$$= -b_1(a_2c_3 - c_2a_3) + b_2(a_1c_3 - c_1a_3) - b_3(a_1c_2 - c_1a_2)$$

Ex: Evaluate  $\begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 5 \\ 3 & 1 & 2 \end{vmatrix}$

$$= -2(4 - 15) + 4(2 - 3) - 1(5 - 2)$$

$$= (-2) \times (-11) + 4 \times (-1) - 1(3)$$

$$= 22 - 4 - 3$$

$$= 15$$



## Minors & cofactors :-

Minor :-

The minor of a given element of determinant is the determinant obtained by deleting the row and column in which the given element stands.

Cofactor :-

If  $M_{ij}$  represents the minor of the element belonging to  $i^{\text{th}}$  row and  $j^{\text{th}}$  column then the cofactor of that element is given by  $C_{ij} = (-1)^{i+j} M_{ij}$

Ex :- 
$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 5 \\ 3 & 1 & 9 \end{vmatrix}$$

Find its minors & cofactors.

Minors -

$$M_{11} = 4 \times 9 - 5 \times 1 = 31$$

$$M_{12} = 18 - 15 = 3$$

$$M_{13} = 2 - 12 = -10$$

$$M_{21} = 18 - 1 = 17$$

$$M_{22} = 9 - 3 = 6$$

$$M_{23} = 1 - 6 = -5$$

$$M_{31} = 10 - 4 = 6$$

$$M_{32} = 5 - 2 = 3$$

$$M_{33} = 4 - 4 = 0$$

cofactors -

$$C_{11} = (-1)^{1+1} 31 = 31$$

$$C_{21} = (-1)^{2+1} (17) = -17$$

$$C_{31} = (-1)^{3+1} (6) = 6$$

$$C_{12} = (-1)^{1+2} 3 = -3$$

$$C_{22} = (-1)^{2+2} (6) = 6$$

$$C_{32} = (-1)^{3+2} (3) = -3$$

$$C_{13} = (-1)^{1+3} (-10) = -10$$

$$C_{23} = (-1)^{2+3} (-5) = 5$$

$$C_{33} = (-1)^{3+3} (0) = 0$$



## Properties of determinants

(1) The value of determinant remains not changed, if the rows & columns are interchange.

Ex :-

$$\text{If } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

then,

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(2) If any two rows or columns of a determinant be interchanged, the value of determinant is changed in sign only.

Ex :-

$$\text{Let, } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_1 = - \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Then  $\Delta_1 = -\Delta$

(3) If all the elements of a row or column are 0, then the value of the determinant is 0.

Ex :-

$$\Delta = \begin{vmatrix} 0 & 0 & 0 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$



(4) if all the elements of any row or column are multiplied by same number, then the determinant is multiplied by that number.

Ex :-

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_1 = \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= K \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= K\Delta$$

(5) if all the elements of a row or column are identical to the element of any other row, then the determinant value is 0.

Ex :-

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

(6) if each element of any row or column is expressed as a sum of two terms then the determinant can be expressed as the sum of two determinants.

Ex :-

$$\Delta = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1+a & y_2+b & y_3+c \\ z_1 & z_2 & z_3 \end{vmatrix}$$



then,

$$\Delta = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} + \begin{vmatrix} x_1 & x_2 & x_3 \\ a & b & c \\ z_1 & z_2 & z_3 \end{vmatrix}$$

row & column operation :-

(7) The value of determinant remains not changed under a column  $(C_i)$  operation of the form ;

$$C_i \rightarrow C_i + \alpha C_j + \beta C_k \quad (j, k \neq i) \text{ and row operation of}$$

$$\text{the form } R_i \rightarrow R_i + \alpha R_j + \beta R_k \quad (j, k \neq i)$$

Ex :

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$R_1 \rightarrow R_1 + \alpha R_2 + \beta R_3$$

$$= \begin{vmatrix} a_1 + \alpha a_2 + \beta a_3 & b_1 + \alpha b_2 + \beta b_3 & c_1 + \alpha c_2 + \beta c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(8) Factor theorem :-

If each element of a matrix  $A$  is a polynomial in  $x$  and if determinant value is 0 for  $x=k$ , then  $x-k$  is factor of determinant.



(2) Multiplication of two determinants

(i) If  $A = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ ,  $B = \begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix}$

then,  $A \times B = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix}$

$$= \begin{vmatrix} a_1 c_1 + b_1 c_2 & a_1 d_1 + b_1 d_2 \\ a_2 c_1 + b_2 c_2 & a_2 d_1 + b_2 d_2 \end{vmatrix}$$

(ii) If  $A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ ,  $B = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$

then,

$$A \times B = \begin{vmatrix} a_1 x_1 + b_1 x_2 + c_1 x_3 & a_1 y_1 + b_1 y_2 + c_1 y_3 & a_1 z_1 + b_1 z_2 + c_1 z_3 \\ a_2 x_1 + b_2 x_2 + c_2 x_3 & a_2 y_1 + b_2 y_2 + c_2 y_3 & a_2 z_1 + b_2 z_2 + c_2 z_3 \\ a_3 x_1 + b_3 x_2 + c_3 x_3 & a_3 y_1 + b_3 y_2 + c_3 y_3 & a_3 z_1 + b_3 z_2 + c_3 z_3 \end{vmatrix}$$

Ex: prove that  $\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix}$

Soln  
L.H.S:  $\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} \times \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}$

$$= \begin{vmatrix} 0+c^2+b^2 & 0+0+ab & 0+ac+0 \\ 0+0+ab & c^2+0+a^2 & bc+0+0 \\ 0+ac+0 & bc+0+0 & b^2+a^2+0 \end{vmatrix} = \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix} = \text{R.H.S}$$



# Nature of system of linear equations and Cramer's rule →

(1) Nature of system of linear eqn with two variables :-

(A) Consistent equation :-

$$\text{Let, } a_1x + b_1y + c_1 = 0 \text{ — (I)}$$

$$a_2x + b_2y + c_2 = 0 \text{ — (II)}$$

(i) Unique and definite solution

→ A system of linear eqn is said to be consistent, if it has at least one unique sol<sup>n</sup>.

$$\text{then, } \frac{a_1}{a_2} \neq \frac{b_1}{b_2} \text{ (unique sol<sup>n</sup>)}$$

(ii) Infinite solution

→ A system of linear eqns is said to be consistent and dependent if it has infinite sol<sup>s</sup>.

$$\text{then, } \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \text{ (infinite sol<sup>n</sup>)}$$

(B) Inconsistent equations (No solution) :-

→ A system of linear eqns is said to be inconsistent, if it has no solution.

$$\text{then, } \frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2} \text{ (No sol<sup>n</sup>)}$$

System of linear equations (Cramer's rule) in two variables →

$$\text{Let, } a_1x + b_1y + c_1 = 0 \text{ — (I)}$$

$$a_2x + b_2y + c_2 = 0 \text{ — (II)}$$

$$\text{where, } \frac{a_1}{a_2} \neq \frac{b_1}{b_2} \text{ (unique sol<sup>n</sup>)}$$



(i) By cross-multiplication,

we get,

$$\frac{x}{(b_1c_2 - b_2c_1)} = \frac{y}{(c_1a_2 - c_2a_1)} = \frac{1}{(a_1b_2 - a_2b_1)}$$

$$\text{or, } \frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$\text{or, } x = \frac{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$\text{Ex: } x + y = 5 \quad \text{--- (i)}$$

$$3x - 2y = 7 \quad \text{--- (ii)}$$

$$a_1 = 1, \quad b_1 = 1 \quad \& \quad c_1 = -5$$

$$a_2 = 3, \quad b_2 = -2 \quad \& \quad c_2 = -7$$

$$x = \frac{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} 1 & -5 \\ -2 & -7 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix}} = \frac{-7 + 10}{-2 - 3} = \frac{3}{-5}$$

$$\Rightarrow \boxed{x = -3/5}$$



$$y = \frac{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} -5 & 1 \\ -7 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix}} = \frac{-15 - (-7)}{-2 - 3} = \frac{-15 + 7}{-2 - 3} = \frac{-8}{-5} = \frac{8}{5}$$

$$\rightarrow \boxed{y = 8/5}$$

system of linear eqns in three variable  $\rightarrow$

$$a_1x + b_1y + c_1z = d_1 \text{ --- (I)}$$

$$a_2x + b_2y + c_2z = d_2 \text{ --- (II)}$$

$$a_3x + b_3y + c_3z = d_3 \text{ --- (III)}$$

Then,

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$\Delta z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

if  $\Delta \neq 0$ , then,

$$x = \frac{\Delta x}{\Delta}, \quad y = \frac{\Delta y}{\Delta}, \quad z = \frac{\Delta z}{\Delta}$$



## Nature of system of linear eq<sup>n</sup> with three variables :-

Let us consider a system of linear eq<sup>n</sup>s be,

$$a_1x + b_1y + c_1z = d_1 \quad \text{--- (I)}$$

$$a_2x + b_2y + c_2z = d_2 \quad \text{--- (II)}$$

$$a_3x + b_3y + c_3z = d_3 \quad \text{--- (III)}$$

Then,  $x = \frac{\Delta_x}{\Delta}$ ,  $y = \frac{\Delta_y}{\Delta}$ ,  $z = \frac{\Delta_z}{\Delta}$

~~where  $\Delta_x, \Delta_y, \Delta_z$  means,~~  
Note :

(i) if  $\Delta \neq 0$  and atleast one of  $\Delta_x, \Delta_y, \Delta_z \neq 0$ , then the given system of eq<sup>n</sup>s is consistent and has unique non trivial sol<sup>n</sup>.

(ii) if  $\Delta \neq 0$  and  $\Delta_x = \Delta_y = \Delta_z = 0$ , then the given system of eq<sup>n</sup>s is consistent and has trivial sol<sup>n</sup> only.

(iii) if  $\Delta = \Delta_x = \Delta_y = \Delta_z = 0$ , then the given system of eq<sup>n</sup>s is consistent and has infinite sol<sup>n</sup>s.

(iv) if  $\Delta = 0$  but atleast one of  $\Delta_x, \Delta_y, \Delta_z$  is not zero then the eq<sup>n</sup>s are inconsistent and have no solution.

Ex :  $x + 2y + z = 7 \quad \text{--- (I)}$

$$2x + 4y + 5z = 8 \quad \text{--- (II)}$$

$$3x + y + 9z = 6 \quad \text{--- (III)}$$

$$\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 5 \\ 3 & 1 & 9 \end{vmatrix}$$

$$= 1(36-5) - 2(18-45) + 1(2-12) \\ = 31 - 6 - 10 = 15$$

$$\Delta_x = \begin{vmatrix} 7 & 2 & 1 \\ 8 & 4 & 5 \\ 6 & 1 & 9 \end{vmatrix}$$

$$= 7(36-5) - 2(72-45) + 1(8-24) \\ = 217 - 84 - 16 = 117$$



$$\Delta_y = \begin{vmatrix} 1 & 7 & 1 \\ 2 & 8 & 5 \\ 3 & 6 & 9 \end{vmatrix}$$

$$= 1(72-30) - 7(18-15) + 1(12-24)$$

$$= 42 - 21 - 12$$

$$= 9$$

$$\Delta_z = \begin{vmatrix} 1 & 2 & 7 \\ 2 & 4 & 8 \\ 3 & 1 & 6 \end{vmatrix}$$

$$= 1(24-8) - 2(12-24) + 7(2-12)$$

$$= 16 + 24 - 70$$

$$= -30$$

$$x = \frac{\Delta_x}{\Delta} = \frac{117}{15}, \quad y = \frac{\Delta_y}{\Delta} = \frac{9}{15}, \quad z = \frac{\Delta_z}{\Delta} = \frac{-30}{15} = -2$$

$$\therefore x = 117/15, \quad y = 9/15 \neq z = -2.$$

Homogeneous system of linear equations :-

$$\text{Let, } a_1x + b_1y + c_1z = 0 \quad \text{--- (I)}$$

$$a_2x + b_2y + c_2z = 0 \quad \text{--- (II)}$$

$$a_3x + b_3y + c_3z = 0 \quad \text{--- (III)}$$

be a system of homogeneous linear eq<sup>n</sup>s in three variables,

$$\Rightarrow \Delta_1 = \Delta_2 = \Delta_3 = 0$$

Therefore, the system always possesses atleast one solution

$x=0, y=0, z=0$ , which is called trivial solution, i.e., this system is always consistent.

Note :

If the system of equations  $a_1x + b_1y + c_1z = 0$ ,  $a_2x + b_2y + c_2z = 0$ ,  $a_3x + b_3y + c_3z = 0$  is always consistent then,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$



# INTEGRAL CALCULUS

Integral calculus is the part of calculus that studies the calculation of integrals to find quantities such as area, volume, displacement & total change.

Integrals are two types :-

1. Indefinite integrals
2. Definite integrals.

Indefinite integrals :-

→ If  $f$  and  $F$  are functions of  $x$  such that  $F'(x) = f(x)$  then the function  $F$  is called an integral of  $f(x)$  w.r.t.  $x$ .

Then,

$$\boxed{\int f(x) dx = F(x) + c}$$

where, ' $c$ ' is called the constant of integration &  $f(x)$  is the integral.

$$\Leftrightarrow \frac{d}{dx}(f(x) + c) = f(x)$$

→ It is denoted as  $\int$  sign.

$$\text{Ex : } F(x) = 4x^3 + 9x + 11$$

$$F'(x) = \frac{d}{dx}(4x^3) + \frac{d}{dx}(9x) + \frac{d}{dx}(11)$$

$$= 12x^2 + 9 + 0$$

$$= 12x^2 + 9$$

$$\text{Let, } f'(x) = f(x)$$

$$\int f(x) dx$$

$$= \int (12x^2 + 9) dx$$



$$= \int 12x^2 dx + \int 9 dx$$

$$= 12 \int x^2 dx + 9 \int dx$$

$$= 12 \cdot \frac{x^{2+1}}{2+1} + 9x + c$$

$$= 4x^3 + 9x + c$$

Properties of indefinite integrals : —

$$(1) \int a f(x) dx$$

$$= a \int f(x) dx, \text{ where 'a' is constant}$$

$$(2) \int (f(x) + g(x)) dx$$

$$= \int f(x) dx + \int g(x) dx$$

$$(3) \text{ If } \int f(y) dy = F(y) + c,$$

$$\text{then } \int f(ax+b) dx = \frac{1}{a} F(ax+b) + c, a \neq 0$$

Standard Results : —

$$(1) \int (ax+b)^n dx$$

$$= \frac{(ax+b)^{n+1}}{a(n+1)} + c; n \neq -1$$

$$(2) \int \frac{dx}{ax+b}$$

$$= \frac{1}{a} e^{\frac{ax+b}{a}} + c \log |ax+b| + c$$

$$(3) \int a^{px+q} dx$$

$$= \frac{1}{p} \frac{a^{px+q}}{\log a} + c, (a > 0)$$



$$(4) \int a^{px+q} dx$$

$$= \frac{1}{-p} \frac{a^{px+q}}{\log a} + c$$

$$(4) \int e^{ax+b} dx$$

$$= \frac{1}{a} e^{ax+b} + c$$

$$(5) \int \sin(ax+b) dx$$

$$= -\frac{1}{a} \cos(ax+b) + c$$

$$(6) \int \cos(ax+b) dx$$

$$= \frac{1}{a} \sin(ax+b) + c$$

$$(7) \int \tan(ax+b) dx$$

$$= \frac{1}{a} \log |\sec(ax+b)| + c$$

$$(8) \int \cot(ax+b) dx$$

$$= \frac{1}{a} \log |\sin(ax+b)| + c$$

$$(9) \int \sec^2(ax+b) dx$$

$$= \frac{1}{a} \tan(ax+b) + c$$

$$(10) \int \operatorname{cosec}^2(ax+b) dx$$

$$= -\frac{1}{a} \cot(ax+b) + c$$

$$(11) \int \operatorname{cosec}(ax+b) \cdot \cot(ax+b) dx$$

$$= -\frac{1}{a} \operatorname{cosec}(ax+b) + c$$



$$(12) \int \sec(ax+b) \cdot \tan(ax+b) dx$$

$$= \frac{1}{a} \sec(ax+b) + c$$

$$(13) \int \sec x dx$$

$$= \log |\sec x + \tan x| + c$$

$$= \log \left| \tan \left( \frac{\pi}{2} + \frac{x}{2} \right) \right| + c$$

$$(14) \int \operatorname{cosec} x dx$$

$$= \log |\operatorname{cosec} x - \cot x| + c$$

$$= \log \left| \tan \frac{x}{2} \right| + c$$

$$(15) \int \frac{dx}{\sqrt{a^2 - x^2}}$$

$$= \sin^{-1} \frac{x}{a} + c$$

$$(16) \int \frac{dx}{a^2 + x^2}$$

$$= \frac{1}{a} \tan^{-1} \frac{x}{a} + c$$

$$(17) \int \frac{dx}{x\sqrt{x^2 - a^2}}$$

$$= \frac{1}{a} \sec^{-1} \frac{x}{a} + c$$

$$(18) \int \frac{dx}{\sqrt{x^2 + a^2}}$$

$$= \log [x + \sqrt{x^2 + a^2}] + c$$

$$(19) \int \frac{dx}{\sqrt{x^2 - a^2}}$$

$$= \log [x + \sqrt{x^2 - a^2}] + c$$



$$(20) \int \frac{dx}{a^2 - x^2}$$

$$= \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c$$

$$(21) \int \frac{dx}{x^2 - a^2}$$

$$= \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c$$

$$(22) \int \sqrt{a^2 - x^2} dx$$

$$= \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + c$$

$$(23) \int \sqrt{x^2 + a^2} dx$$

$$= \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log (x + \sqrt{x^2 + a^2}) + c$$

$$(24) \int \sqrt{x^2 - a^2} dx$$

$$= \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log (x + \sqrt{x^2 - a^2}) + c$$

$$(25) \int e^{ax} \cdot \sin bx dx$$

$$= \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + c$$

$$(26) \int e^{ax} \cdot \cos bx dx$$

$$= \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) + c$$



Examples based on standard results :-

$$(1) \int x^4 dx$$

$$= \frac{x^{4+1}}{4+1} + C$$

$$= \frac{x^5}{5} + C$$

$$\boxed{\int x^n dx = \frac{x^{n+1}}{n+1} + C}$$

$$(2) \int (3x^2 - 10x + 7) dx$$

$$= \int 3x^2 dx - \int 10x dx + \int 7 dx$$

$$= 3 \int x^2 dx - 10 \int x dx + 7 \int dx$$

$$= 3 \times \frac{x^3}{3} - 10 \times \frac{x^2}{2} + 7x + C$$

$$= x^3 - 5x^2 + 7x + C$$

$$(3) \int \frac{dx}{(x+7)}$$

$$= \log |x+7| + C$$

$$(4) \int \frac{dx}{ax-7}$$

$$= \frac{1}{a} \log |ax-7| + C$$

$$\boxed{\int \frac{dx}{ax+b} = \frac{1}{a} \log |ax+b| + C}$$

$$(5) \int e^{5x} dx$$

$$= \frac{1}{5} e^{5x} + C$$

$$= \frac{e^{5x}}{5} + C$$

$$\boxed{\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + C}$$

$$(6) \int a^{qx} dx$$

$$= \frac{1}{q} \frac{a^{qx}}{\log a} + C$$

$$\boxed{\int a^{px+q} dx = \frac{1}{p} \frac{a^{px+q}}{\log a} + C}$$



$$(7) \int \cos 4x dx$$

$$= \frac{1}{4} \sin 4x + c$$

$$\int \cos(ax+b) dx = \frac{1}{a} \sin(ax+b) + c$$

$$(8) \int \sin \frac{x}{3} dx$$

$$= \frac{-1}{1/3} \cos \frac{x}{3} + c$$

$$= -3 \cos \frac{x}{3} + c$$

$$\int \sin(ax+b) dx = -\frac{1}{a} \cos(ax+b) + c$$

$$(9) \int \tan 9x dx$$

$$= \frac{1}{9} \log |\sec 9x| + c$$

$$\int \tan(ax+b) dx = \frac{1}{a} \log |\sec(ax+b)| + c$$

$$(10) \int x^2 \cot x^3 dx$$

$$= \frac{1}{3} \int 3x^2 \cot x^3 dx$$

$$= \frac{1}{3} \int \cot x^3 (3x^2) dx$$

$$= \frac{1}{3} \int \frac{\cos x^3}{\sin x^3} \cdot 3x^2 dx$$

$$= \frac{1}{3} \int \frac{dt}{t}$$

$$= \frac{1}{3} \log |t| + c$$

$$= \frac{1}{3} \log |\sin x^3| + c$$

$$\text{Let, } \sin x^3 = t$$

$$\frac{d(\sin x^3)}{dx} = \frac{dt}{dx}$$

$$\cos x^3 \cdot 3x^2 dx = dt$$

$$\int \frac{dt}{t} = \log |t| + c$$

$$(11) \int \sec x dx$$

$$= \int \frac{\sec x (\sec x + \tan x)}{(\sec x + \tan x)} dx$$



$$= \int \frac{(\sec^2 x + \sec x \tan x) dx}{(\sec x + \tan x)}$$

$$\text{Let, } \sec x + \tan x = t$$

$$\frac{dt}{dx} \sec^2 x + \sec x \tan x dx = dt$$

$$= \int \frac{dt}{t}$$

$$= \log |t| + C$$

$$= \log |\sec x + \tan x| + C$$

$$(12) \int \frac{dx}{1 + \cos x}$$

$$= \int \frac{(1 - \cos x)}{(1 + \cos x)(1 - \cos x)} dx$$

$$= \int \frac{(1 - \cos x)}{(1 - \cos^2 x)} dx$$

$$= \int \frac{(1 - \cos x)}{\sin^2 x} dx$$

$$= \int \left( \frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x} \right) dx$$

$$= \int \left( \operatorname{cosec}^2 x - \frac{\cos x}{\sin x} \times \frac{1}{\sin x} \right) dx$$

$$= \int (\operatorname{cosec}^2 x - \cot x \cdot \operatorname{cosec} x) dx$$

$$= \int \operatorname{cosec}^2 x - \int \cot x \cdot \operatorname{cosec} x dx$$

$$= -\cot x - (-\operatorname{cosec} x) + C$$

$$= -\cot x + \operatorname{cosec} x + C$$



$$(13) \int \operatorname{cosec} x \, dx$$

$$= \log \left| \tan \frac{x}{2} \right| + c$$

$$\boxed{\int \operatorname{cosec} x \, dx = \log \left| \tan \frac{x}{2} \right| + c}$$

$$(14) \int \frac{dx}{\sqrt{9-x^2}}$$

$$= \int \frac{dx}{\sqrt{(3)^2-x^2}}$$

$$= \sin^{-1} \frac{x}{3} + c$$

$$\boxed{\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a} + c}$$

$$(15) \int \frac{dx}{16+x^2}$$

$$= \int \frac{dx}{\sqrt{(4)^2+x^2}}$$

$$= \frac{1}{4} \tan^{-1} \frac{x}{4} + c$$

$$\boxed{\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c}$$

$$(16) \int \frac{dx}{x\sqrt{9x^2-25}}$$

$$= \int \frac{3 \, dx}{3x\sqrt{(3x)^2-(5)^2}}$$

$$= \int \frac{dt}{t\sqrt{t^2-5^2}}$$

$$= \frac{1}{5} \sec^2 \left( \frac{t}{5} \right) + c$$

$$= \frac{1}{5} \sec^2 \left( \frac{3x}{5} \right) + c$$

$$\text{Let, } 3x = t$$

$$\frac{d(3x)}{dx} = \frac{dt}{dx}$$

$$3 \, dx = dt$$

$$\boxed{\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} \sec^{-1} \frac{x}{a} + c}$$

$$(17) \int \frac{(x+5)}{\sqrt{7-4x-x^2}} \, dx$$

$$\text{Let, } 7-4x-x^2 = t^2$$



$$\frac{d}{dx} (7 - 4x - x^2) = \frac{d}{dx} t^2$$

$$\Rightarrow (0 - 4 - 2x) \frac{dx}{dx} = 2t \frac{dt}{dx}$$

$$\Rightarrow (-4 - 2x) dx = 2t dt$$

$$\Rightarrow 2(-2 - x) dx = 2t dt$$

$$\Rightarrow (-2 - x) dx = t dt$$

$$= - \int \frac{(-x-2)}{\sqrt{7-4x-x^2}} dx$$

$$= - \int \frac{\{(-x-2)-3\}}{\sqrt{7-4x-x^2}} dx$$

$$= - \left\{ \int \frac{(-x-2)}{\sqrt{7-4x-x^2}} dx - \int \frac{3 dx}{\sqrt{7-4x-x^2}} \right\}$$

$$= - \left\{ \int \frac{t dt}{\sqrt{t^2}} - \int \frac{3 dx}{\sqrt{7-4x-x^2}} \right\}$$

$$= - \left\{ \int \frac{t dt}{t} - 3 \int \frac{dx}{\sqrt{7-(4x+x^2)}} \right\}$$

$$= - \left\{ \int \frac{t dt}{t} - 3 \int \frac{dx}{\sqrt{7-(2 \cdot x \cdot 2 + (2)^2 + (x)^2 - (2)^2)}} \right\}$$

$$= - \left\{ \int dt - 3 \int \frac{dx}{11 - (x+2)^2} \right\}$$

$$= - \left\{ t - 3 \int \frac{dx}{(\sqrt{11})^2 - (x+2)^2} \right\}$$



$$= - \left\{ \sqrt{7-4x-x^2} - 3x \sin^{-1} \left( \frac{x+2}{\sqrt{11}} \right) + c \right\}$$

$$= -\sqrt{7-4x-x^2} + 3 \sin^{-1} \left( \frac{x+2}{\sqrt{11}} \right) + c \quad (\text{Ans})$$

$$(18) \int \frac{dx}{(x^2-1)}$$

$$= \int \frac{dx}{x^2-1^2}$$

$$\boxed{\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c}$$

$$= \frac{1}{2 \times 1} \log \left| \frac{x-1}{x+1} \right| + c$$

$$= \frac{1}{2} \log \left| \frac{x-1}{x+1} \right| + c$$

$$(19) \int \frac{dx}{1-x^2}$$

$$= \int \frac{dx}{1^2-x^2}$$

$$\boxed{\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c}$$

$$= \frac{1}{2} \log \left| \frac{1+x}{1-x} \right| + c$$

$$(20) \int \frac{dx}{\sqrt{x^2-1}}$$

$$= \int \frac{dx}{\sqrt{x^2-1^2}}$$

$$\boxed{\int \frac{dx}{\sqrt{x^2-a^2}} = \log [x + \sqrt{x^2-a^2}] + c}$$

$$= \log [x + \sqrt{x^2-1}] + c$$

$$(21) \int \frac{dx}{\sqrt{x^2+1}}$$

$$= \int \frac{dx}{\sqrt{x^2+1^2}}$$

$$\boxed{\int \frac{dx}{\sqrt{x^2+a^2}} = \log |x + \sqrt{x^2+a^2}| + c}$$

$$= \log |x + \sqrt{x^2+1}| + c$$



$$(22) \int \frac{dx}{\sqrt{4x^2+25}}$$

$$= \int \frac{dx}{\sqrt{(2x)^2+(5)^2}}$$

$$\text{Let, } 2x = t$$

$$= \int \frac{dt}{2\sqrt{t^2+(5)^2}}$$

$$\frac{d(2x)}{dx} = \frac{dt}{dx}$$

$$2 dx = dt$$

$$dx = \frac{dt}{2}$$

$$= \frac{1}{2} \int \frac{dt}{\sqrt{t^2+5^2}}$$

$$= \frac{1}{2} \log |t + \sqrt{t^2+5^2}| + C$$

$$= \frac{1}{2} \log |2x + \sqrt{4x^2+25}| + C$$

$$(23) \int \sqrt{4-x^2} dx$$

$$= \int \sqrt{(2)^2-x^2} dx$$

$$\boxed{\int \sqrt{a^2-x^2} dx = \frac{x}{2} \sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C}$$

$$= \frac{x}{2} \sqrt{4-x^2} + \frac{4}{2} \sin^{-1} \frac{x}{2} + C$$

$$= \frac{x}{2} \sqrt{4-x^2} + 2 \sin^{-1} \frac{x}{2} + C$$

$$(24) \int \sqrt{x^2-9} dx$$

$$= \int \sqrt{x^2-(3)^2} dx$$

$$\boxed{\int \sqrt{x^2-a^2} dx = \frac{x}{2} \sqrt{x^2-a^2} - \frac{a^2}{2} \log |x + \sqrt{x^2-a^2}| + C}$$

$$= \frac{x}{2} \sqrt{x^2-9} - \frac{9}{2} \log (x + \sqrt{x^2-9}) + C$$

$$(25) \int \sqrt{7x^2+11} dx$$

$$\text{Let, } \sqrt{7x} = t$$

$$\frac{d(\sqrt{7x})}{dx} = \frac{dt}{dx}$$

$$\sqrt{7} dx = dt$$

$$= \int \frac{\sqrt{(\sqrt{7x})^2+11}}{\sqrt{7}} dt$$

$$dx = \frac{dt}{\sqrt{7}}$$



$$= \int \frac{\sqrt{t^2 + 11}}{\sqrt{7}} dt$$

$$= \frac{1}{\sqrt{7}} \int \sqrt{t^2 + (11)^2} dt$$

$$= \frac{1}{\sqrt{7}} \left[ \frac{t}{2} \sqrt{t^2 + a^2} + \frac{a^2}{2} \log(t + \sqrt{t^2 + a^2}) + C \right]$$

$$= \frac{t}{2\sqrt{7}} \sqrt{t^2 + a^2} + \frac{a^2}{2\sqrt{7}} \log |t + \sqrt{t^2 + a^2}| + C$$

$$= \frac{\sqrt{7}x}{2\sqrt{7}} \sqrt{7x^2 + 11} + \frac{11}{2\sqrt{7}} \log |\sqrt{7}x + \sqrt{7x^2 + 11}| + C$$

$$= \frac{x}{2} \sqrt{7x^2 + 11} + \frac{11}{2\sqrt{7}} \log |\sqrt{7}x + \sqrt{7x^2 + 11}| + C$$

Important Methods of Integration :-

Rule 1 : By changing variables

Rule 2 : Integration by parts

Rule 3 : Integration by partial fractions

Rule - 1 : By changing variables

If  $f(x)$  is a continuous differentiable function, then to evaluate integral of the form,

$$\int \phi(f(x)) f'(x) dx$$

$$\text{Let, } f(x) = t$$

$$\frac{d}{dx} f(x) = \frac{dt}{dx}$$

$$f'(x) dx = dt$$



Hence,  $I = \int \phi(f(x)) f'(x) dx$

$$= \int \phi(t) dt$$

$$\Rightarrow \int [Q(x)]^n Q'(x) dx$$

or

$$\int \frac{Q'(x)}{\sqrt{Q(x)}} dx$$

or

$$\int \frac{Q'(x)}{[Q(x)]^n} dx$$

we put,  $Q(x) = t$

Examples :-

$$(1) \quad I = \int \frac{\cos(\log x)}{x} dx$$

$$I = \int \cos t dt$$

$$= \sin t + c$$

$$= \sin(\log x) + c$$

$$(2) \quad I = \int \frac{\sin(\log x)}{x} dx$$

$$I = \int \sin t dt$$

$$= -\cos t + c$$

$$= -\cos(\log x) + c$$

Let,  $\log x = t$

$$\frac{d(\log x)}{dx} = \frac{dt}{dx}$$

$$\frac{1}{x} = \frac{dt}{dx}$$

$$\frac{1}{x} dx = dt$$

Let,  $\log x = t$

$$\frac{d(\log x)}{dx} = \frac{dt}{dx}$$

$$\frac{1}{x} = \frac{dt}{dx}$$

$$\frac{1}{x} dx = dt$$



$$(3) I = \int \frac{(1+\log x)^3}{x} dx$$

$$\text{Let, } 1+\log x = t$$

$$I = \int (t)^3 dt$$

$$\frac{d}{dx} (1+\log x) = \frac{dt}{dx}$$

$$= \frac{t^{3+1}}{3+1} + C$$

$$\frac{1}{x} dx = dt$$

$$= \frac{t^4}{4} + C$$

$$\therefore I = \frac{(1+\log x)^4}{4} + C$$

$$(4) I = \int \frac{(1+\log x)^5}{x} dx$$

$$\text{Let, } 1+\log x = t$$

$$I = \int (t)^5 dt$$

$$\frac{d}{dx} (1+\log x) = \frac{dt}{dx}$$

$$= \frac{t^{5+1}}{5+1} + C$$

$$\frac{1}{x} dx = dt$$

$$= \frac{t^6}{6} + C$$

$$\therefore I = \frac{(1+\log x)^6}{6} + C$$

$$(5) I = \int e^x \csc e^x dx$$

$$\text{Let, } e^x = t$$

$$I = \int \csc t dt$$

$$\frac{d}{dx} e^x = \frac{dt}{dx}$$

$$= \sin t + C$$

$$e^x dx = dt$$

$$= \sin e^x + C$$



$$(6) I = \int e^x \sin e^x dx$$

$$\text{Let, } e^x = t$$

$$I = \int \sin t dt$$

$$= -\cos t + C$$

$$= -\cos e^x + C$$

$$\frac{d e^x}{d x} = \frac{d t}{d x}$$

$$e^x dx = dt$$

Rule - 2 : Integration By parts

If  $u$  and  $v$  are differentiable functions of  $x$  then,

$$\int u \cdot v dx = u \int v dx - \int [u' \cdot \int v dx] dx$$

$$\text{where, } u' = \frac{du}{dx}$$

$u = 1^{\text{st}}$  function  $\nabla$

$v = 2^{\text{nd}}$  function

→ For this reason it is called integration by parts.

Uses →

**I L A T E**

where, I = inverse function

Ex:  $\sin^{-1}x$ ,  $\cos^{-1}x$ ,  $\tan^{-1}x$  ... etc

L = logarithm function

Ex:  $\log x$ ,  $\ln x$

A = Algebraic function

Ex:  $x$ ,  $2x$ ,  $x^2$ ,  $x^3$  ... etc

T = Trigonometric function

Ex:  $\sin x$ ,  $\cos x$  ... etc

E = Exponential function

Ex:  $e^x$ ,  $e^{2x}$  ... etc



Examples :-

(1) Evaluate  $\int \log x \, dx$

$$= \int \log x \cdot 1 \, dx$$

$$= \log x \int 1 \, dx - \int \left[ \frac{d(\log x)}{dx} \cdot \int 1 \, dx \right] dx$$

$$= \log x \times x - \int \left[ \frac{1}{x} \times x \right] dx$$

$$= x \log x - \int 1 \, dx$$

$$= x \log x - x + c$$

(2)  $\int \log_2 x \, dx$

$$= \int \log_2 x \cdot 1 \, dx$$

$$= \log_2 x \int 1 \, dx - \int \left[ \frac{d(\log_2 x)}{dx} \cdot \int 1 \, dx \right] dx$$

$$= \log_2 x \times x - \int \left[ \frac{1}{x \ln 2} \times x \right] dx$$

$$= x \log_2 x - \int 1 \, dx$$

$$= x \log_2 x - x + c$$

(3)  $I = \int \sec^3 \theta \, d\theta$

$$I = \int \sec \theta \cdot \sec^2 \theta \, d\theta$$

$$I = \sec \theta \int \sec^2 \theta \, d\theta - \int \left[ \frac{d(\sec \theta)}{d\theta} \cdot \int \sec^2 \theta \, d\theta \right] d\theta$$

$$I = \sec \theta \cdot \tan \theta - \int \left[ (\sec \theta \cdot \tan \theta) \cdot \tan \theta \right] d\theta$$

$$I = \sec \theta \tan \theta - \int (\sec \theta \cdot \tan^2 \theta) d\theta$$

$$I = \sec \theta \tan \theta - \int (\sec^2 \theta - 1) \sec \theta d\theta$$



$$I = \sec\theta \tan\theta - \int (\sec^3\theta - \sec\theta) d\theta$$

$$I = \sec\theta \cdot \tan\theta - \int \sec^3\theta d\theta + \int \sec\theta d\theta$$

$$I = \sec\theta \cdot \tan\theta - I + \log |\sec\theta + \tan\theta| + C$$

$$I + I = \sec\theta \cdot \tan\theta + \log |\sec\theta + \tan\theta| + C$$

$$2I = \sec\theta \cdot \tan\theta + \log |\sec\theta + \tan\theta| + C$$

$$I = \frac{1}{2} \sec\theta \cdot \tan\theta + \frac{1}{2} \log |\sec\theta + \tan\theta| + C$$

$$(4) I = \int e^x \sin x dx$$

$$I = \int \sin x \cdot e^x dx$$

$$I = \sin x \int e^x dx - \int \left[ \frac{d(\sin x)}{dx} \cdot \int e^x dx \right] dx$$

$$I = \sin x \cdot e^x - \int (\cos x \cdot e^x) dx$$

$$I = e^x \sin x - \left[ \cos x \int e^x dx - \int \left[ \frac{d(\cos x)}{dx} \cdot \int e^x dx \right] dx \right]$$

$$I = e^x \sin x - \left[ \cos x \cdot e^x - \int ((-\sin x) \cdot e^x) dx \right]$$

$$I = e^x \sin x - e^x \cos x + \int (\sin x \cdot e^x) dx$$

$$I = e^x \sin x - e^x \cos x - \int e^x \sin x dx$$

$$I = e^x \sin x - e^x \cos x - I + C$$

$$2I = e^x \sin x - e^x \cos x + C$$

$$I = \frac{1}{2} [e^x \sin x - e^x \cos x] + C$$



# INTEGRAL CALCULUS

Integral calculus is the part of calculus that studies the calculation of integrals to find quantities such as area, volume, displacement & total change.

Integral are two types:-

1. Indefinite integral

2. Definite integral.

Indefinite integral:-

→ if  $f$  and  $F$  are functions of  $x$  such that  $f'(x) = f(x)$  then the function  $F$  is called an integral of  $f(x)$  w.r.t.  $x$ .

Then

$$\int f(x) dx = F(x) + C$$

where ' $C$ ' is called the constant of integration &  $f(x)$  is the integrand.

$$\Leftrightarrow \frac{d}{dx} (f(x) + C) = f(x)$$

→ It is denoted as  $\int$  sign

$$\text{Ex: } F(x) = 4x^3 + 9x + 11$$

$$F'(x) = \frac{d}{dx} (4x^3) + \frac{d}{dx} (9x) + \frac{d}{dx} (11)$$

$$= 12x^2 + 9 + 0$$

$$= 12x^2 + 9$$

$$\text{Let, } f'(x) = f(x)$$

$$\int f(x) dx$$



part of  
the calculation  
quantities such  
as  $\Delta$  total

u:

functions of  $x$   
then the  
an integral

the constant  
is the

$\int$  sign

$f(x) + \frac{d}{dx} (II)$

$$\begin{aligned} &= \int (12x^2 + 9) dx \\ &= \int 12x^2 dx + \int 9 dx \\ &= 12 \cdot \frac{x^{2+1}}{2+1} + 9x + C \\ &= 4x^3 + 9x + C \end{aligned}$$

Properties of indefinite integrals:-

$$1. \int a f(x) dx$$

$$= a \int f(x) dx, \text{ where 'a' is constant}$$

$$2. \int (f(x) + g(x)) dx$$

$$= \int f(x) dx + \int g(x) dx$$

$$3. \text{ If } \int f(y) dy = F(y) + C,$$

$$\text{then } \int f(ax+b) dx = 1/a \cdot F(ax+b) + C, a \neq 0$$

Standard results:-

$$1. \int (ax+b)^n dx$$

$$= \frac{(ax+b)^{n+1}}{a(n+1)} + C; n \neq -1$$

$$2. \int \frac{dx}{ax+b}$$

$$= 1/a \log |ax+b| + C$$

$$3. \int a^{px+q} dx$$

$$= 1/p \cdot \frac{a^{px+q}}{\log a} + C, (a > 0)$$

$$4. \int e^{ax+b} dx$$

$$= 1/a e^{ax+b} + C$$



$$5. \int \sin(ax+b) dx$$

$$= -1/a \cos(ax+b) + C$$

$$6. \int \cos(ax+b) dx$$

$$= 1/a \sin(ax+b) + C$$

$$7. \int \tan(ax+b) dx$$

$$= 1/a \log |\sec(ax+b)| + C$$

$$8. \int \cot(ax+b) dx$$

$$= 1/a \log |\sin(ax+b)| + C$$

$$9. \int \sec^2(ax+b) dx$$

$$= 1/a \tan(ax+b) + C$$

$$10. \int \operatorname{cosec}^2(ax+b) dx$$

$$= -1/a \cot(ax+b) + C$$

$$11. \int \operatorname{cosec}(ax+b) \cdot \cot(ax+b) dx$$

$$= -1/a \operatorname{cosec}(ax+b) + C$$

$$12. \int \sec(ax+b) \cdot \tan(ax+b) dx$$

$$= 1/a \sec(ax+b) + C$$

$$13. \int \sec x dx$$

$$= \log |\sec x + \tan x| + C$$

$$= \log \left| \tan \left( \pi/2 + x/2 \right) \right| + C$$

$$14. \int \operatorname{cosec} x dx$$

$$= \log |\operatorname{cosec} x - \cot x| + C$$

$$= \log |\tan x/2| + C$$

$$15. \int \frac{dx}{\sqrt{a^2 - x^2}}$$

$$= \sin^{-1} x/a + C$$

$$16. \int \frac{dx}{a^2 + x^2}$$

$$= 1/a \tan^{-1} x/a$$

$$17. \int \frac{dx}{x \sqrt{x^2 - a^2}}$$

$$= 1/a \sec^{-1} x/a$$

$$18. \int \frac{dx}{\sqrt{x^2 + a^2}}$$

$$= \log [x + \sqrt{x^2 + a^2}]$$

$$19. \int \frac{dx}{\sqrt{x^2 - a^2}}$$

$$= \log [x + \sqrt{x^2 - a^2}]$$

$$20. \int \frac{dx}{a^2 - x^2}$$

$$= 1/2a \log \left| \frac{a+x}{a-x} \right|$$

$$21. \int \frac{dx}{x^2 - a^2}$$

$$= 1/2a \log \left| \frac{x-a}{x+a} \right|$$

$$22. \int \sqrt{a^2 - x^2} dx$$

$$= x/2 \sqrt{a^2 - x^2} + a^2/2 \sin^{-1} x/a$$

$$23. \int \sqrt{x^2 + a^2} dx$$

$$= x/2 \sqrt{x^2 + a^2} + a^2/2 \log [x + \sqrt{x^2 + a^2}]$$



$$15. \int \frac{dx}{\sqrt{a^2 - x^2}}$$

$$= \sin^{-1} x/a + c$$

$$16. \int \frac{dx}{a^2 + x^2}$$

$$= 1/a \tan^{-1} x/a + c$$

$$17. \int \frac{dx}{x\sqrt{x^2 - a^2}}$$

$$= 1/a \sec^{-1} x/a + c$$

$$18. \int \frac{dx}{\sqrt{x^2 + a^2}}$$

$$= \log [x + \sqrt{x^2 + a^2}] + c$$

$$19. \int \frac{dx}{\sqrt{x^2 - a^2}}$$

$$= \log [x + \sqrt{x^2 - a^2}] + c$$

$$20. \int \frac{dx}{a^2 - x^2}$$

$$= 1/2a \log \left| \frac{a+x}{a-x} \right| + c$$

$$21. \int \frac{dx}{x^2 - a^2}$$

$$= 1/2a \log \left| \frac{x-a}{x+a} \right| + c$$

$$22. \int \sqrt{a^2 - x^2} dx$$

$$= x/2 \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} x/a + c$$

$$23. \int \sqrt{x^2 + a^2} dx$$

$$= x/2 \sqrt{x^2 + a^2} + \frac{a^2}{2} \log (x + \sqrt{x^2 + a^2}) + c$$



$$24. \int \sqrt{a^2 - x^2} dx$$

$$= \frac{x}{2} \sqrt{a^2 - x^2} - \frac{a^2}{2} \log (x + \sqrt{a^2 - x^2}) + C$$

$$25. \int e^{ax} \sin bx dx$$

$$= \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + C$$

$$26. \int e^{ax} \cos bx dx$$

$$= \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) + C$$

Examples based on standard result:

$$1. \int x^4 dx$$

$$= \frac{x^{4+1}}{4+1} + C$$

$$= \frac{x^5}{5} + C$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$2. \int (3x^2 - 10x + 7) dx$$

$$= \int 3x^2 dx - \int 10x dx + \int 7 dx$$

$$= 3 \int x^2 dx - 10 \int x dx + 7 \int dx$$

$$= \cancel{3}x^{\cancel{3}}^3 / \cancel{3} - \cancel{10}x^{\cancel{2}}^2 / \cancel{2} + 7x + C$$

$$= x^3 - 5x^2 + 7x + C$$

$$3. \int \frac{dx}{(x+7)}$$

$$= \log |x+7| + C$$



$$4. \int \frac{dx}{9x-7}$$

$$= \frac{1}{9} \log |9x-7| + C$$

$$5. \int e^{5x} dx$$

$$= \frac{1}{5} e^{5x} + C$$

$$= \frac{e^{5x}}{5} + C$$

$$6. \int a^{9x} dx$$

$$= \frac{1}{9} \cdot \frac{a^{9x}}{\log a} + C$$

$$7. \int \cos 4x dx$$

$$= \frac{1}{4} \sin 4x + C$$

$$8. \int \sin x/3 dx$$

$$= -1/1/3 \cos x/3 + C$$

$$= -3 \cos x/3 + C$$

$$9. \int \tan 9x dx$$

$$= \frac{1}{9} \log |\sec 9x| + C$$

$$10. \int x^2 \cot x^3 dx$$

$$= \frac{1}{3} \int 3x^2 \cot x^3 dx$$

$$= \frac{1}{3} \int \cot x^3 (3x^2) dx$$

$$= \frac{1}{3} \int \frac{\cos x^3}{\sin x^3} \cdot 3x^2 dx$$

$$= \frac{1}{3} \int \frac{dt}{t}$$

$$= \frac{1}{3} \log |t| + C$$

$$= \frac{1}{3} \log |\sin x^3| + C$$

$$\int \frac{dx}{ax+b} = \frac{1}{a} \log |ax+b| + C$$

$$\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + C$$

$$\int a^{px+q} dx = \frac{1}{p} \cdot \frac{a^{px+q}}{\log a} + C$$

$$\int \cos(ax+b) dx$$

$$= \frac{1}{a} \sin(ax+b) + C$$

$$\int \sin(ax+b) dx$$

$$= -\frac{1}{a} \cos(ax+b) + C$$

$$\int \tan(ax+b) dx$$

$$= \frac{1}{a} \log |\sec(ax+b)| + C$$

$$\text{Let, } \sin x^3 = t$$

$$\Rightarrow \frac{d}{dx} (\sin x^3) = \frac{dt}{dx}$$

$$\Rightarrow \cos x^3 \cdot 3x^2 dx = dt$$

$$\int \frac{dt}{t} = \log |t| + C$$



$$11. \int \sec x \, dx$$

$$= \int \frac{\sec x (\sec x + \tan x)}{(\sec x + \tan x)} \, dx$$

Let,

$$\sec x + \tan x = t$$

$$\Rightarrow \sec^2 x + \sec x \tan x \, dx = dt$$

$$= \int \frac{(\sec^2 x + \sec x \tan x)}{(\sec x + \tan x)} \, dx$$

$$= \int \frac{dt}{t}$$

$$= \log |t| + C$$

$$= \log |\sec x + \tan x| + C$$

$$12. \int \frac{dx}{1 + \cos x}$$

$$= \int \frac{(1 - \cos x)}{(1 + \cos x)(1 - \cos x)} \, dx$$

$$= \int \frac{(1 - \cos x)}{(1 - \cos^2 x)} \, dx$$

$$= \int \frac{(1 - \cos x)}{(1 - \cos^2 x)} \, dx$$

$$= \int \frac{(1 - \cos x)}{\sin^2 x} \, dx$$

$$= \int \left( \frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x} \right) \, dx$$

$$= \int \left( \operatorname{cosec}^2 x - \frac{\cos x}{\sin x} \times \frac{1}{\sin x} \right) \, dx$$

$$= \int (\operatorname{cosec}^2 x - \cot x \cdot \operatorname{cosec} x) \, dx$$

$$= \int \operatorname{cosec}^2 x - \int \cot x \cdot \operatorname{cosec} x \, dx$$

$$= -\cot x - (-\operatorname{cosec} x) + C$$

$$= -\cot x + \operatorname{cosec} x + C$$



$$13. \int \operatorname{cosec} x \, dx$$

$$= \log |\tan x/2| + c$$

$$\int \operatorname{cosec} x \, dx = \log |\tan x/2| + c$$

$$14. \int \frac{dx}{\sqrt{9-x^2}}$$

$$= \int \frac{dn}{\sqrt{(3)^2-x^2}}$$

$$= \sin^{-1} x/3 + c$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a} + c$$

$$15. \int \frac{dx}{\sqrt{(4)^2+x^2}}$$

$$15. \int \frac{dx}{16+x^2}$$

$$= \int \frac{dx}{\sqrt{(4)^2+x^2}}$$

$$= 1/4 \tan^{-1} x/4 + c$$

$$\int \frac{dx}{a^2+x^2} = 1/a \tan^{-1} \frac{x}{a} + c$$

$$16. \int \frac{dx}{x\sqrt{9x^2-25}}$$

$$= \int \frac{3 \, dx}{3x\sqrt{(3x)^2-(5)^2}}$$

$$= \int \frac{dt}{t\sqrt{t^2-5^2}}$$

$$= 1/5 \sec^{-1} (t/5) + c$$

$$= 1/5 \sec^{-1} (3x/5) + c$$

$$\text{Let, } 3x = t$$

$$\Rightarrow \frac{d}{dx} (3x) = \frac{dt}{dx}$$

$$\Rightarrow 3 \, dx = dt$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = 1/a \sec^{-1} x/a + c$$

$$17. \int \frac{(x+5)}{\sqrt{7-4x-x^2}} \, dx$$

$$= \text{Let, } 7-4x-x^2 = t^2$$



$$\frac{d}{dx} (7 - 4x - x^2) = \frac{d}{dx} x^2$$

$$\Rightarrow (0 - 4 - 2x) \frac{dx}{dx} = 2t \frac{dt}{dx}$$

$$\Rightarrow (-4 - 2x) dx = 2t dt$$

$$\Rightarrow \cancel{2}(-2-x) dx = \cancel{2} t dt$$

$$\Rightarrow (-2-x) dx = t dt$$

$$= - \int \frac{(x+2)}{\sqrt{7-4x-x^2}} dx$$

$$= - \int \frac{(-x-2) = 3}{\sqrt{7-4x-x^2}} dx$$

$$= - \left\{ \int \frac{(-x-2)}{\sqrt{7-4x-x^2}} dx - \int \frac{3 dx}{\sqrt{7-4x-x^2}} \right\}$$

$$= - \left\{ \int \frac{t dt}{\sqrt{t^2}} - \int \frac{3 dx}{\sqrt{7-4x-x^2}} \right\}$$

$$= - \left\{ \int \frac{t dt}{t} - 3 \int \frac{dx}{\sqrt{7-(4x+x^2)}} \right\}$$

$$= - \left\{ \int \frac{t dt}{t} - 3 \int \frac{dx}{\sqrt{7-(2 \cdot x \cdot 2 + (2)^2 + (x)^2 - (2)^2)}} \right\}$$

$$= - \left\{ \int dt - 3 \int \frac{dx}{11 - (x+2)^2} \right\}$$

$$= - \left\{ t - 3 \int \frac{dx}{(\sqrt{11})^2 - (x+2)^2} \right\}$$

$$= - \left\{ \sqrt{7-4x-x^2} - 3x \sin^{-1} \left( \frac{x+2}{\sqrt{11}} \right) + C \right\}$$

$$= - \sqrt{7-4x-x^2} + 3 \sin^{-1} \left( \frac{x+2}{\sqrt{11}} \right) + C$$

$$19. \int \frac{dx}{1-x^2}$$

$$= \int \frac{dx}{1-x^2}$$

$$= \frac{1}{2}$$

$$20. \int \frac{dx}{\sqrt{x}}$$

$$= \int \frac{dx}{\sqrt{x}}$$

$$= \log$$

$$21. \int \frac{dx}{\sqrt{x}}$$

$$= \int$$

$$= \log$$

$$220. \int$$

$$= \int$$

$$= \int$$

$$= \frac{1}{2}$$

$$=$$

$$=$$



$$19. \int \frac{dx}{1-x^2}$$

$$= \int \frac{dx}{1^2 - x^2}$$

$$= \frac{1}{2} \log \left| \frac{1+x}{1-x} \right| + C$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + C$$

$$20. \int \frac{dx}{\sqrt{x^2-1}}$$

$$= \int \frac{dx}{\sqrt{x^2 - 1^2}}$$

$$= \log [x + \sqrt{x^2 - 1}] + C$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \log [x + \sqrt{x^2 - a^2}] + C$$

$$21. \int \frac{dx}{\sqrt{x^2+1}}$$

$$= \int \frac{dx}{\sqrt{x^2 + 1^2}}$$

$$= \log |x + \sqrt{x^2 + 1}| + C$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \log |x + \sqrt{x^2 + a^2}| + C$$

$$22. \int \frac{dx}{\sqrt{4x^2+25}}$$

$$= \int \frac{dx}{\sqrt{(2x)^2 + (5)^2}}$$

$$= \int \frac{dt}{2\sqrt{(t)^2 + (5)^2}}$$

$$= \frac{1}{2} \int \frac{dt}{\sqrt{t^2 + 5^2}}$$

$$= \frac{1}{2} \log |t + \sqrt{t^2 + 5^2}| + C$$

$$= \frac{1}{2} \log |2x + \sqrt{4x^2 + 25}| + C$$

$$\text{Let, } 2x = t$$

$$\Rightarrow \frac{d}{dx}(2x) = \frac{dt}{dx}$$

$$\Rightarrow 2dx = dt$$

$$\Rightarrow dx = \frac{dt}{2}$$



$$\begin{aligned}
 23. \int \sqrt{4-x^2} dx & \quad \int \sqrt{a^2-x^2} dx \\
 & = \frac{x}{2} \sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C \\
 & = \int \sqrt{(2)^2-x^2} dx \\
 & = \frac{x}{2} \sqrt{4-x^2} + \frac{4}{2} \sin^{-1} \frac{x}{2} + C \\
 & = \frac{x}{2} \sqrt{4-x^2} + 2 \sin^{-1} \frac{x}{2} + C
 \end{aligned}$$

$$\begin{aligned}
 24. \int \sqrt{x^2-9} dx & \quad \int \sqrt{x^2-a^2} dx \\
 & = \int \sqrt{x^2-(3)^2} dx = \frac{x}{2} \sqrt{x^2-a^2} - \frac{a^2}{2} \log |x + \sqrt{x^2-a^2}| + C \\
 & = \frac{x}{2} \sqrt{x^2-9} - \frac{9}{2} \log (x + \sqrt{x^2-9}) + C
 \end{aligned}$$

$$\begin{aligned}
 25. \int \sqrt{7x^2+11} dx & \\
 & = \int \frac{\sqrt{(\sqrt{7}x)^2+11}}{\sqrt{7}} dt \\
 & = \int \frac{\sqrt{t^2+11}}{\sqrt{7}} dt \\
 & = \frac{1}{\sqrt{7}} \int \sqrt{t^2+(\sqrt{11})^2} dt \\
 & = \frac{1}{\sqrt{7}} \left[ \frac{t}{2} \sqrt{t^2+a^2} + \frac{a^2}{2} \log (t + \sqrt{t^2+a^2}) + C \right] \\
 & = \frac{t}{2\sqrt{7}} \sqrt{t^2+a^2} + \frac{a^2}{2\sqrt{7}} \log |t + \sqrt{t^2+a^2}| + C \\
 & = \frac{\sqrt{7}x}{2\sqrt{7}} \sqrt{7x^2+11} + \frac{11}{2\sqrt{7}} \log |\sqrt{7}x + \sqrt{7x^2+11}| + C \\
 & = \frac{x}{2} \sqrt{7x^2+11} + \frac{11}{2\sqrt{7}} \log |\sqrt{7}x + \sqrt{7x^2+11}| + C
 \end{aligned}$$

Important methods of integration:

Rule 1: By changing variables

Rule 2: Integration by parts

Rule 3: Integration by partial fraction



Rule 1 On changing variables

$$\int f(x) f'(x) dx$$

$$\text{Let, } f(x) = t$$

$$\frac{d}{dx} f(x) = \frac{dt}{dx}$$

$$f'(x) dx = dt$$

Hence,

$$I = \int \phi(f(x)) f'(x) dx$$

$$= \int \phi(t) dt$$

$$\Rightarrow \int [Q(x)]^n Q'(x) dx$$

or

$$\int \frac{Q'(x)}{\sqrt{Q(x)}} dx$$

or

$$\int \frac{Q'(x)}{[Q(x)]^n} dx$$

We put,

$$Q(x) = t$$

Examples:-

$$\text{Ex. } I = \int \frac{\cos(\log x)}{x} dx$$

$$I = \int \cos t dt$$

$$= \sin t + C$$

$$= \sin(\log x) + C$$

$$\text{Let } \log x = t$$

$$\Rightarrow \frac{d}{dx}(\log x) = \frac{dt}{dx}$$

$$\Rightarrow \frac{1}{x} = \frac{dt}{dx}$$

$$\Rightarrow \frac{1}{x} dx = dt$$



$$2. I = \int \frac{\sin(\log x)}{x} dx$$

$$I = \int \sin t dt$$

$$= -\cos t + C$$

$$= -\cos(\log x) + C$$

$$\text{Let, } \log x = t$$

$$\Rightarrow \frac{d}{dx}(\log x) = \frac{dt}{dx}$$

$$\Rightarrow \frac{1}{x} = \frac{dt}{dx}$$

$$\Rightarrow \frac{1}{x} dx = dt$$

$$3. I = \int \frac{(1 + \log x)^3}{x} dx$$

$$I = \int (t)^3 dt$$

$$= \frac{t^{3+1}}{3+1} + C$$

$$= \frac{t^4}{4} + C$$

$$\therefore I = \frac{(1 + \log x)^4}{4} + C$$

$$4. I = \int \frac{(1 + \log x)^5}{x} dx$$

$$I = \int (t)^5 dt$$

$$= \frac{t^{5+1}}{5+1} + C$$

$$= \frac{t^6}{6} + C$$

$$\therefore I = \frac{(1 + \log x)^6}{6} + C$$

$$\text{Let, } 1 + \log x = t$$

$$\Rightarrow \frac{d}{dx}(1 + \log x) = \frac{dt}{dx}$$

$$\Rightarrow \frac{1}{x} dx = dt$$

$$\text{Let, } 1 + \log x = t$$

$$\Rightarrow \frac{d}{dx}(1 + \log x) = \frac{dt}{dx}$$

$$\Rightarrow \frac{1}{x} dx = dt$$

$$5. I = \int e^x \cos e^x dx$$

$$I = \int \cos t dt$$

$$= \sin t + C$$

$$= \sin e^x + C$$

$$\text{Let, } e^x = t$$

$$\Rightarrow \frac{d}{dx} e^x = \frac{dt}{dx}$$

$$\Rightarrow e^x dx = dt$$



Let,  $\log x = t$   
 $\Rightarrow \frac{d}{dx}(\log x) = \frac{dt}{dx}$

$\Rightarrow \frac{1}{x} = \frac{dt}{dx}$

$\Rightarrow \frac{1}{x} dx = dt$

Let,  $1 + \log x = t$   
 $\Rightarrow \frac{d}{dx}(1 + \log x) = \frac{dt}{dx}$

$\Rightarrow \frac{1}{x} dx = dt$

Let,  $1 + \log x = t$   
 $\Rightarrow \frac{d}{dx}(1 + \log x) = \frac{dt}{dx}$

$\Rightarrow \frac{1}{x} dx = dt$

Let,  $e^x = t$

$\Rightarrow \frac{d}{dx} e^x = \frac{dt}{dx}$

$\Rightarrow e^x dx = dt$

e.  $I = \int e^x \sin e^x dx$

$I = \int \sin t dt$

$= -\cos t + C$

$= -\cos e^x + C$

Let,  $e^x = t$

$\Rightarrow \frac{d}{dx} e^x = \frac{dt}{dx}$

$\Rightarrow e^x dx = dt$

Rule - 2: Integration by parts.

If  $u$  and  $v$  are differentiable function of  $x$  then,

$\int u.v dx = u \int v dx - \int [u' \cdot \int v dx] dx$

where,  $u' = \frac{du}{dx}$

$u = 1^{st}$  function

$v = 2^{nd}$  function

for this reason it is called integration by parts.

Uses  $\rightarrow$

ILATE

where,  $I =$  Inverse function

ex:  $\sin^{-1} x, \cos^{-1} x, \tan^{-1} x \dots$  etc

$L =$  logarithm function.

ex:  $\log x, \ln x$

$A =$  Algebraic function

ex:  $x, 2x, x^2, x^3 \dots$  etc.

$T =$  Trigonometric function.

ex:  $\sin x, \cos x, \tan x \dots$  etc.

$E =$  Exponential function.

ex:  $e^x, e^{2x} \dots$  etc



Examples :-

1. Evaluate  $\int \log x \, dx$

$$= \int \log x \cdot 1 \, dx$$

$$= \log x \int 1 \, dx - \left[ \frac{d}{dx} (\log x) \cdot \int 1 \, dx \right] dx$$

$$= \log x \times x - \int \left[ \frac{1}{x} \times x \right] dx$$

$$= x \log x - \int 1 \, dx$$

$$= x \log x - x + c$$

2.  $\int \log 2x \, dx$

$$= \int \log 2x \cdot 1 \, dx$$

$$= \log 2x \int 1 \, dx - \left[ \frac{d}{dx} (\log 2x) \cdot \int 1 \, dx \right] dx$$

$$= \log 2x \times x - \int \left[ \frac{1}{2x} \times 2 \times x \right] dx$$

$$= x \log 2x - \int 1 \, dx$$

$$= x \log 2x - x + c$$

3.  $I = \int \sec^3 \theta \, d\theta$

$$\Rightarrow I = \int \sec \theta \cdot \sec^2 \theta \, d\theta$$

$$\Rightarrow I = \sec \theta \int \sec^2 \theta \, d\theta - \left[ \frac{d}{d\theta} (\sec \theta) \cdot \int \sec^2 \theta \, d\theta \right] d\theta$$

$$\Rightarrow I = \sec \theta \cdot \tan \theta - \int \left[ (\sec \theta \cdot \tan \theta) \cdot \tan \theta \right] d\theta$$

$$\Rightarrow I = \sec \theta \cdot \tan \theta - \int (\sec \theta \cdot \tan^2 \theta) \, d\theta$$

$$\Rightarrow I = \sec \theta \cdot \tan \theta - \int (\sec^2 \theta - 1) \sec \theta \, d\theta$$

$$\Rightarrow I = \sec \theta \tan \theta - \int (\sec^3 \theta - \sec \theta) \, d\theta$$

$$\Rightarrow I = \sec \theta \cdot \tan \theta - \int \sec^3 \theta \, d\theta + \int \sec \theta \, d\theta$$



$$\Rightarrow I = \sec \theta \cdot \tan \theta - I + \log |\sec \theta + \tan \theta| + C$$

$$\Rightarrow I + I = \sec \theta \cdot \tan \theta + \log |\sec \theta + \tan \theta| + C$$

$$\Rightarrow 2I = \sec \theta \cdot \tan \theta + \log |\sec \theta + \tan \theta| + C$$

$$\Rightarrow I = \frac{1}{2} \sec \theta \cdot \tan \theta + \frac{1}{2} \log |\sec \theta + \tan \theta| + C$$

$$4. I = \int e^x \sin x \, dx$$

$$\Rightarrow I = \int \sin x \cdot e^x \, dx$$

$$\Rightarrow I = \sin x \int e^x \, dx - \int \left[ \frac{d}{dx} (\sin x) \cdot \int e^x \, dx \right] dx$$

$$\Rightarrow I = \sin x \cdot e^x - \int (\cos x \cdot e^x) \, dx$$

$$\Rightarrow I = e^x \sin x - \left[ \cos x \int e^x \, dx - \int \frac{d}{dx} (\cos x) \int e^x \, dx \right]$$

$$\Rightarrow I = e^x \sin x - \left[ \cos x \cdot e^x - \int (-\sin x) \cdot e^x \, dx \right]$$

$$\Rightarrow I = e^x \sin x - e^x \cos x + \int (\sin x) \cdot e^x \, dx$$

$$\Rightarrow I = e^x \sin x - e^x \cos x - I + C$$

$$\Rightarrow I = e^x \sin x - e^x \cos x - I + C$$

$$\Rightarrow 2I = e^x \sin x - e^x \cos x + C$$

$$\Rightarrow I = \frac{1}{2} [e^x \sin x - e^x \cos x] + C$$



Rule :- as - Integration by partial fractions.

A function of the form  $\frac{f(x)}{g(x)}$ , where  $f(x)$  and  $g(x)$  are polynomials is called a rational fraction.

It is called proper rational fraction if the degree of  $f(x)$  is less than the degree of  $g(x)$ , otherwise is called an improper rational fraction. Every proper rational fraction can be expressed as a sum of simpler fractions, called partial fraction.  $A_1, A_2, A_3$  are constant.

| Proper Rational fraction                 | Partial fraction                                                  |
|------------------------------------------|-------------------------------------------------------------------|
| $\frac{Px+q}{(ax+b)(cx+d)}$              | $-\frac{A_1}{ax+b} + \frac{A_2}{cx+d}$                            |
| $= \frac{Px+q}{(ax+b)^2(cx+d)}$          | $-\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(cx+d)}$ |
| $= \frac{Px^2+qx+r}{(ax+b)(cx+d)(ex+f)}$ | $-\frac{A_1}{ax+b} + \frac{A_2}{cx+d} + \frac{A_3}{ex+f}$         |
| $= \frac{Px^2+qx+r}{(ax+b)^2(cx+d)}$     | $-\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(cx+d)}$ |

~~Partial fraction~~

~~$$\frac{A_1}{ax+b} + \frac{A_2}{cx+d}$$~~



partial

$f(x)$ ,  
 $g(x)$  is

partial fraction  
 less than  
 degree of  
 denominator  
 function.  
 can be  
 simpler  
 reaction.

partial fraction

$f)$

$\frac{A_3}{ex+f}$

$\frac{A_3}{(cx+d)}$

$$= \frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(cx+d)}$$

$$= \frac{A_1}{ax+b} + \frac{A_2}{cx+d} + \frac{A_3}{ex+f}$$

$$= \frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(cx+d)}$$

$$Q. I = \int \frac{(3x+1)}{(x+1)(x-2)} dx$$

Solution:-

Hence the integral is a proper rational function. We get,

$$\frac{3x+1}{(x+1)(x-2)} = \frac{A_1}{(x+1)} + \frac{A_2}{(x-2)}$$

$$\Rightarrow \frac{3x+1}{(x+1)(x-2)} = \frac{A_1(x-2) + A_2(x+1)}{(x+1)(x-2)}$$

$$\Rightarrow 3x+1 = A_1x - 2A_1 + A_2x + A_2$$

$$\Rightarrow 3x+1 = x(A_1+A_2) + A_2 - 2A_1$$

$$\Rightarrow 3x = x(A_1+A_2) \text{ or } 1 = A_2 - 2A_1$$

$$\Rightarrow 3 = (A_1 + A_2) \text{ --- (i) or } 1 = A_2 - 2A_1 \text{ --- (ii)}$$

Add both in eqn (i) & (ii)

$$A_1 + A_2 = 3$$

$$-2A_1 + A_2 = 1$$

$$\begin{array}{r} + \quad - \\ \hline 3A_1 = 2 \end{array}$$

$$A_1 = 2/3$$



In equ<sup>n</sup> (i) put the value of  $A_1 = 2$ ,  
we get,

$$A_1 + A_2 = 3$$

$$\Rightarrow 2/3 + A_2 = 3$$

$$\Rightarrow 2/3 - 3 = A_2$$

$$\Rightarrow \frac{2-9}{3} = A_2$$

$$\Rightarrow -7/3 = A_2$$

$$I = \int \frac{3x+1}{(x+1)(x-2)} dx = \int \frac{2/3}{(x+1)} dx + \int \frac{-7/3}{(x-2)} dx$$

$$I = \int \frac{3x+1}{(x+1)(x-2)} dx = 2/3 \int \frac{dx}{(x+1)} - 7/3 \int \frac{dx}{(x-2)}$$

$$I = \int \frac{3x+1}{(x+1)(x-2)} dx = 2/3 \log(x+1) - 7/3 \log(x-2) + C$$

### Definite Integrals.

The definite integral of continuous function  $f(x)$  defined on the closed interval  $[a, b]$ , is given by  
 $\int_a^b f(x) dx = f(x)_a^b = f(b) - f(a)$ ; where  
 $f(x)$  is an anti-derivative of  $f$ , 'a' is called the lower limit of the integral and 'b' is called upper limit of the integral.



1:2  
Some common properties of definite integrals:

$$1. \int_a^b f(x) dx$$

$$= \int_a^b f(t) dt \text{ provided } f \text{ is same}$$

$$2. \int_a^b f(x) dx$$

$$= - \int_b^a f(x) dx$$

$$3. \int_a^b f(x) dx$$

$$= \int_a^c f(x) dx + \int_c^b f(x) dx, \text{ where } c \text{ may}$$

lie inside or outside the interval  $[a, b]$ . This property is to be used when  $f$  is piecewise continuous in  $(a, b)$ .

$$4. \int_{-a}^a f(x) dx$$

$$= \int_0^a f(x) dx + \int_0^a f(-x) dx \begin{cases} 0; & \text{if } f(x) \text{ is odd function} \end{cases}$$

$$= 2 \int_0^a f(x) dx; \text{ if } f(x) \text{ is even function}$$

$$5. \int_a^b f(x) dx$$

$$= \int_a^b f(a+b-x) dx \text{ in particular}$$

$$\int_0^a f(x) dx$$

$$= \int_0^a f(a-x) dx$$



$$6. \int_0^{2a} f(x) dx$$

$$= \int_0^a f(x) dx + \int_a^{2a} f(x) dx$$

$$= \begin{cases} 2 \int_0^a f(x) dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$$

$$7. \int_0^{nT} f(x) dx$$

$$= n \int_0^T f(x) dx, (n \in \mathbb{I}) : \text{where } T \text{ is the period of the function i.e. } f(T+x) = f(x)$$

Ques:-

Evaluate

$$1. \int_0^2 x^3 dx$$

$$\text{Solution:-} \left[ \frac{x^{3+1}}{3+1} \right]_0^2$$

$$= \left[ \frac{x^4}{4} \right]_0^2$$

$$= \frac{1}{4} [2^4 - 0^4]$$

$$= \frac{1}{4} \times 16$$

$$= 4$$

$$2. \int_0^{\pi/4} \sin^4 2t \cos 2t dt$$

Solution:-

$$\text{put } \sin 2t = x$$

$$\frac{d}{dx} \sin 2t =$$

$$\Rightarrow \cos 2t \cdot \frac{d}{dt} (2t)$$

$$\Rightarrow \cos 2t \cdot 2 \frac{dt}{dt}$$

$$\Rightarrow \cos 2t \cdot 2 dt$$

$$\Rightarrow \cos 2t \cdot dt$$

$$\sin 2t = x$$

$$\Rightarrow t = \pi/4$$

$$\Rightarrow \sin 2 \times \pi/4 = x$$

$$\Rightarrow \sin \pi/2 = x$$

$$\Rightarrow 1 = x$$

$$\int_0^1 x^4 \frac{dx}{2}$$

$$= \frac{1}{2} \int_0^1 x^4 dx$$

$$= \frac{1}{2} \left[ \frac{x^5}{5} \right]_0^1$$

$$= \frac{1}{10} \left[ x^5 \right]_0^1$$



$$(2\omega - \alpha) d\alpha$$

$$f(2\omega - \alpha) = f(\alpha)$$

$$f(\omega - \alpha) = -f(\alpha)$$

$\in I$ ) : where  $\pi'$  is the  
the function i.e.  $f(\pi + \pi')$

$$\left[ \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right]_0^2$$

$$2$$

$$0$$

$$\left[ \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right]$$

$$1/4 \times 16$$

$$2 \int_0^{\pi/4} \sin^4 2t \cos 2t dt$$

Solution:-

$$\text{put } \sin 2t = x$$

$$d/dx \sin 2t = \frac{dx}{dt}$$

$$\Rightarrow \cos 2t \cdot \frac{d}{dt}(2t) = \frac{dx}{dt}$$

$$\Rightarrow \cos 2t \cdot 2 \frac{dt}{dt} = \frac{dx}{dt}$$

$$\Rightarrow \cos 2t \cdot 2 dt = dx$$

$$\Rightarrow \cos 2t \cdot dt = \frac{dx}{2}$$

$$\sin 2t = x$$

$$\Rightarrow t = \pi/4$$

$$\Rightarrow \sin 2 \times \pi/4 = x$$

$$\Rightarrow \sin \pi/2 = x$$

$$\Rightarrow 1 = x$$

$$\Rightarrow t = 0$$

$$\Rightarrow \sin 0 = x$$

$$\Rightarrow 0 = x$$

$$\int_0^1 x^4 \frac{dx}{2}$$

$$= \frac{1}{2} \int_0^1 x^4 dx$$

$$= \frac{1}{2} \left[ \frac{x^5}{5} \right]_0^1$$

$$= \frac{1}{10} \left[ x^5 \right]_0^1$$

$$= \frac{1}{10} \left[ (1)^5 - (0)^5 \right]$$

$$= \frac{1}{10} (1 - 0)$$

$$= \frac{1}{10} \neq \underline{\underline{ng}}$$



Example 12

Evaluate  $\int_0^2 f(x) dx$ , if  $f(x) = \begin{cases} x^3 & 0 < x < 1 \\ 2x+1 & 1 \leq x \leq 2 \end{cases}$

Solution :-

Property - 3

$$\int_0^2 f(x) dx$$

$$= \int_0^1 f(x) dx + \int_1^2 f(x) dx$$

$$= \int_0^1 x^3 dx + \int_1^2 (2x+1) dx$$

$$= \left[ \frac{x^4}{4} \right]_0^1 + \int_1^2 2x dx + \int_1^2 1 dx$$

$$= \frac{1}{4} [x^4]_0^1 + 2 \left[ \frac{x^2}{2} \right]_1^2 + [x]_1^2$$

$$= \frac{1}{4} [1^4 - 0^4] + 2 \left[ \frac{x^2}{2} \right]_1^2 + [2 - 1]$$

$$= \frac{1}{4} [1 - 0] + [2^2 - 1^2] + 1$$

$$= \frac{1}{4} \times 1 + [4 - 1] + 1$$

$$= \frac{1}{4} + 3 + 1$$

$$= \frac{1}{4} + 4$$

$$= \frac{17}{4}$$

Example 13 -

$$I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx$$



$$, \text{ if } f(x) = \begin{cases} x^3 & 0 < x < 1 \\ 2x+1 & 1 \leq x \leq 2 \end{cases}$$

Property - 3

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx \quad \dots \dots \dots (i)$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\cos(\pi/2 + 0 - x)}}{\sqrt{\cos(\pi/2 + 0 - x)} + \sqrt{\sin(\pi/2 + 0 - x)}} dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\cos(\pi/2 - x)}}{\sqrt{\cos(\pi/2 - x)} + \sqrt{\sin(\pi/2 - x)}} dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \quad \dots \dots \dots (ii)$$

From equation (i) and (ii) we get,

$$\Rightarrow 2I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx + \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \left( \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} + \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \right) dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \left( \frac{\sqrt{\cos x} + \sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}} \right) dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} 1 dx$$

$$\Rightarrow 2I = [x]_0^{\pi/2}$$

$$\Rightarrow 2I = [\pi/2 - 0]$$

$$\Rightarrow 2I = \pi/2$$

$$\Rightarrow I = \pi/4$$



# Use of wallis Integral formulae.

The formula attributed to the english mathematician John Wallis (1616-1703) is based on the reduction formula. We will study following two case.

## Case - 1

If  $m, n$  are both positive integers, then

$$\int_0^{\pi/2} \sin^m x \cos^n x dx = \frac{(m-1)(m-3) \dots (1 \text{ or } 2) (n-1)(n-3) \dots (1 \text{ or } 2)}{(m+n)(m+n-2) \dots (1 \text{ or } 2)} \cdot \frac{\pi}{2}$$

if  $m$  &  $n$  both even integer

$$\frac{(m-1)(m-3) \dots (1 \text{ or } 2) (n-1)(n-3) \dots (1 \text{ or } 2)}{(m+n)(m+n-2) \dots (1 \text{ or } 2)} \cdot \frac{\pi}{2}$$

or 1; otherwise

## Case - 2

If  $n$  is a positive integer,

$$\int_0^{\pi/2} \sin^n x dx$$

$$\int_0^{\pi/2} \cos^n x dx = \frac{(n-1)(n-3)(n-5) \dots 3 \cdot 1}{n(n-2)(n-4) \dots 4 \cdot 2} \cdot \frac{\pi}{2}; \text{ if } n \text{ is even.}$$

$$\frac{(n-1)(n-3)(n-5) \dots 3 \cdot 1}{n(n-2)(n-4) \dots 4 \cdot 2}$$

$$\int_0^{\pi/2} \sin^4 x \cdot \cos^6 x dx$$

Solution :-

$$\int_0^{\pi/2} \sin^4 x \cdot \cos^6 x dx$$

Using wallis formula

$$\Rightarrow \int_0^{\pi/2} \sin^4 x \cdot \cos^6 x dx =$$

$$\Rightarrow \int_0^{\pi/2} \sin^4 x \cdot \cos^6 x dx = \frac{(4-1)(4-3)}{(4+6)} \cdot \frac{\pi}{2}$$

$$\Rightarrow \int_0^{\pi/2} \sin^4 x \cdot \cos^6 x dx = \frac{3 \times 2}{10 \times 8} \cdot \frac{\pi}{2}$$

$$\Rightarrow \int_0^{\pi/2} \sin^4 x \cdot \cos^6 x dx =$$

$$\Rightarrow \int_0^{\pi/2} \sin^4 x \cdot \cos^6 x dx =$$

$$\Rightarrow \int_0^{\pi/2} \sin^4 x \cdot \cos^6 x dx =$$

$$\Rightarrow \int_0^{\pi/2} \sin^4 x \cdot \cos^6 x dx =$$

$$\Rightarrow \int_0^{\pi/2} \sin^4 x \cdot \cos^6 x dx =$$



# Integral formulas.

Attributed to the mathematician John Wallis, proof is based on formula. We will have two cases.

both positive integers  $m, n \dots \dots \dots +\infty$

$$\frac{(n-1)(n-3)(n-5) \dots (1 \text{ or } 2)}{(n-2)(n-4)(n-6) \dots (1 \text{ or } 2)} \cdot \frac{\pi}{2}$$

$m$  &  $n$  both even integers

$$\frac{(n-1)(n-3) \dots (1 \text{ or } 2)}{(n-2) \dots (1 \text{ or } 2)} \cdot 1; \text{ otherwise}$$

positive integers,

$$\frac{1}{2} \cdot \frac{\pi}{2}; \text{ if } n \text{ is even.}$$

$$\begin{cases} \frac{(n-1)(n-3)(n-5) \dots 4 \cdot 2}{n(n-2)(n-4) \dots 3 \cdot 1} \cdot 1; & \text{if } n \text{ is odd.} \end{cases}$$

Q.  $\int_0^{\pi/2} \sin^4 x \cdot \cos^6 x \, dx$

Solution :-

$$\int_0^{\pi/2} \sin^4 x \cdot \cos^6 x \, dx \quad m=4, n=6$$

Using Wallis's formula we get,

$$\Rightarrow \int_0^{\pi/2} \sin^4 x \cdot \cos^6 x \, dx = \frac{(4-1)(4-3)(6-1)(6-3)(6-5)}{(4+6)(4+6-2)(4+6-4)(4+6-6)(4+6-8)} \cdot \frac{\pi}{2}$$

$$\Rightarrow \int_0^{\pi/2} \sin^4 x \cdot \cos^6 x \, dx = \frac{(4-1)(4-3)(6-1)(6-3)(6-5)}{(4+6)(4+6-2)(4+6-4)(4+6-6)(4+6-8)} \cdot \frac{\pi}{2}$$

$$\Rightarrow \int_0^{\pi/2} \sin^4 x \cdot \cos^6 x \, dx = \frac{3 \times 1 \times 5 \times 3 \times 1}{10 \times 8 \times 6 \times 4 \times 2} \cdot \frac{\pi}{2}$$

$$\Rightarrow \int_0^{\pi/2} \sin^4 x \cdot \cos^6 x \, dx = \frac{45}{3840} \cdot \frac{\pi}{2}$$

$$\Rightarrow \int_0^{\pi/2} \sin^4 x \cdot \cos^6 x \, dx = \frac{45\pi}{7680}$$

$$\Rightarrow \int_0^{\pi/2} \sin^4 x \cdot \cos^6 x \, dx = \frac{3\pi}{512}$$



$$\text{Q2. } \int_0^{\pi/2} \sin^7 x \cdot \cos^4 x \cdot dx$$

Solution :-

$$m = 7, n = 4$$

Using wallis's formula we get,

$$\Rightarrow \int_0^{\pi/2} \sin^7 x \cdot \cos^4 x \cdot dx = \frac{(m-1)(m-3)(m-5)(n-1)(n-3)}{(m+n)(m+n-2)(m+n-4)(m+n-6)(m+n-8)(m+n-10)} \cdot 1$$

$$\Rightarrow \int_0^{\pi/2} \sin^7 x \cdot \cos^4 x \cdot dx = \frac{(7-1)(7-3)(7-5)(4-1)(4-3)}{(7+4)(7+4-2)(7+4-4)(7+4-6)(7+4-8)(7+4-10)} \cdot 1$$

$$\Rightarrow \int_0^{\pi/2} \sin^7 x \cdot \cos^4 x \cdot dx = \frac{6 \times 4 \times 2 \times 3 \times 1}{11 \times 9 \times 7 \times 5 \times 3 \times 1} \cdot 1$$

$$\Rightarrow \int_0^{\pi/2} \sin^7 x \cdot \cos^4 x \cdot dx = \frac{144}{10395} \times 1$$

$$\Rightarrow \int_0^{\pi/2} \sin^7 x \cdot \cos^4 x \cdot dx = \frac{144}{10395}$$

$$\text{Q3. } \int_0^{\pi/2} \sin^2 x \cdot dx$$

$$n = 2$$

Using wallis's integral formula we get,

$$\Rightarrow \int_0^{\pi/2} \cos^2 x \cdot dx = \frac{(n-1)}{n} \cdot \pi/2$$

$$\Rightarrow \int_0^{\pi/2} \cos^2 x \cdot dx = \frac{2-1}{2} \cdot \pi/2$$

$$\Rightarrow \int_0^{\pi/2} \cos^2 x \cdot dx = \frac{1}{2} \times \pi/2$$

$$\Rightarrow \int_0^{\pi/2} \cos^2 x \cdot dx = \pi/4$$



$$\text{Q4. } \int_0^{\pi/2} \sin^7 x \, dx$$

Solution :-

$$n = 7$$

Using wallis's integral formula we get,

$$\Rightarrow \int_0^{\pi/2} \sin^7 x \, dx = \frac{(n-1)(n-3)(n-5)}{n(n-2)(n-4)(n-6)} \cdot 1$$

$$\Rightarrow \int_0^{\pi/2} \sin^7 x \, dx = \frac{(7-1)(7-3)(7-5)}{7(7-2)(7-4)(7-6)} \cdot 1$$

$$\Rightarrow \int_0^{\pi/2} \sin^7 x \, dx = \frac{6 \times 4 \times 2}{7 \times 5 \times 3 \times 1} \cdot 1$$

$$\Rightarrow \int_0^{\pi/2} \sin^7 x \, dx = \frac{48}{105} \times 1$$

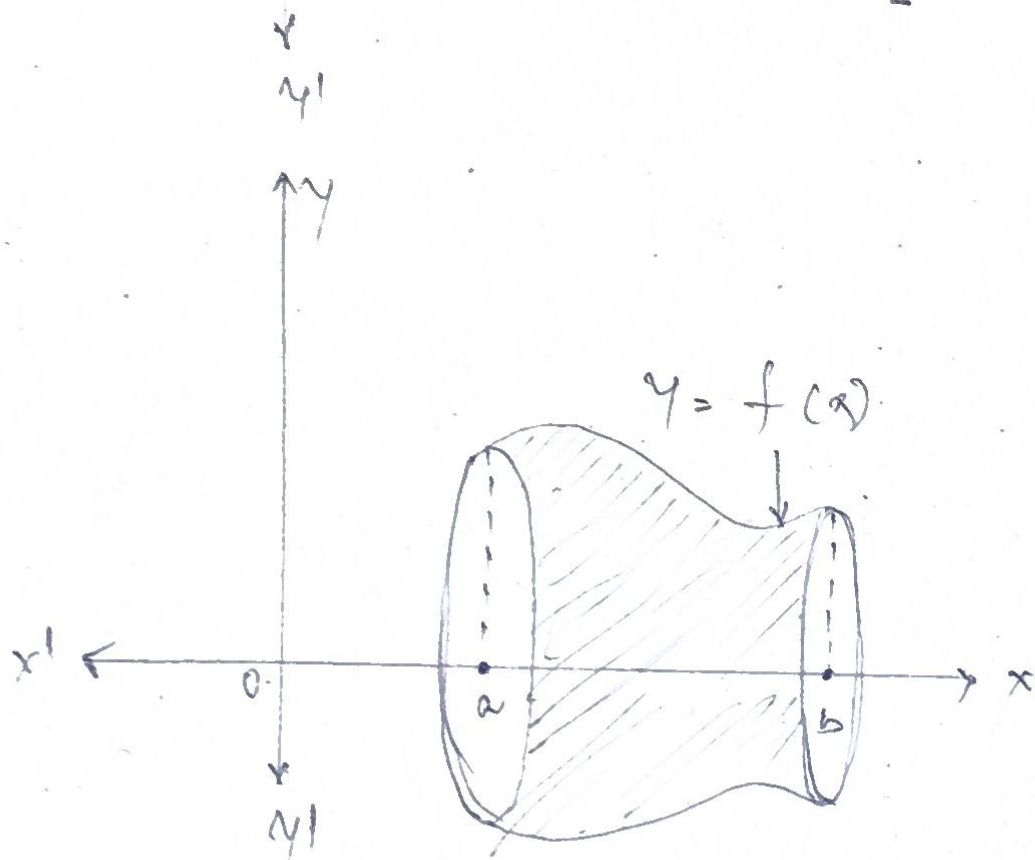
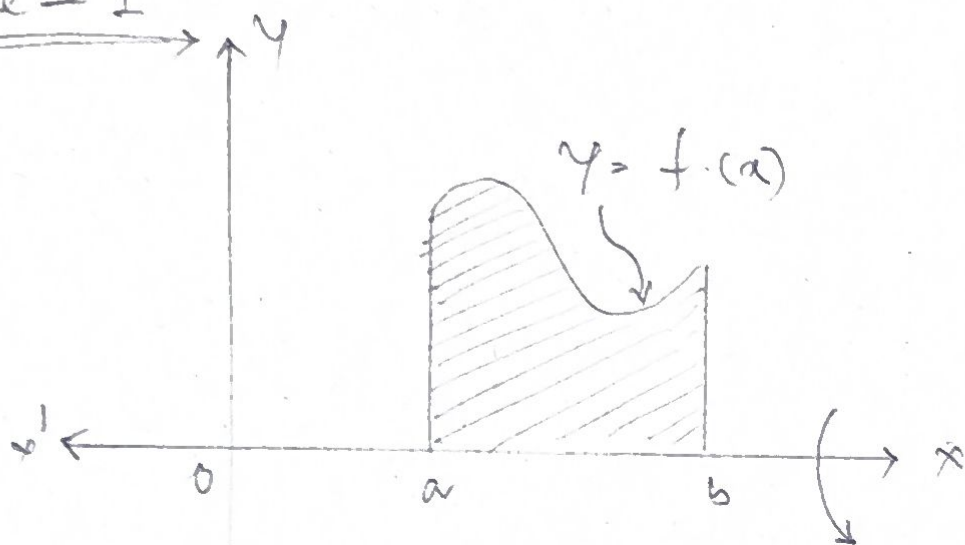
$$\Rightarrow \int_0^{\pi/2} \sin^7 x \, dx = \frac{48}{105}$$



Volume of a solid formed by  
 Revolution of an area about  $x$ -axis  
 (using Disk method)

To find the volume of a solid  
 formed by revolution for  $\phi$  in two  
 cases arise:-

Case - 1



Revolution about  $x$ -axis :-

defn:- "The volume of a solid formed  
 by revolution of an area  
 about  $x$ -axis bounded by the  
 curve  $y = f(x)$  the ordinate  
 $x=a$ ,  $x=b$  then



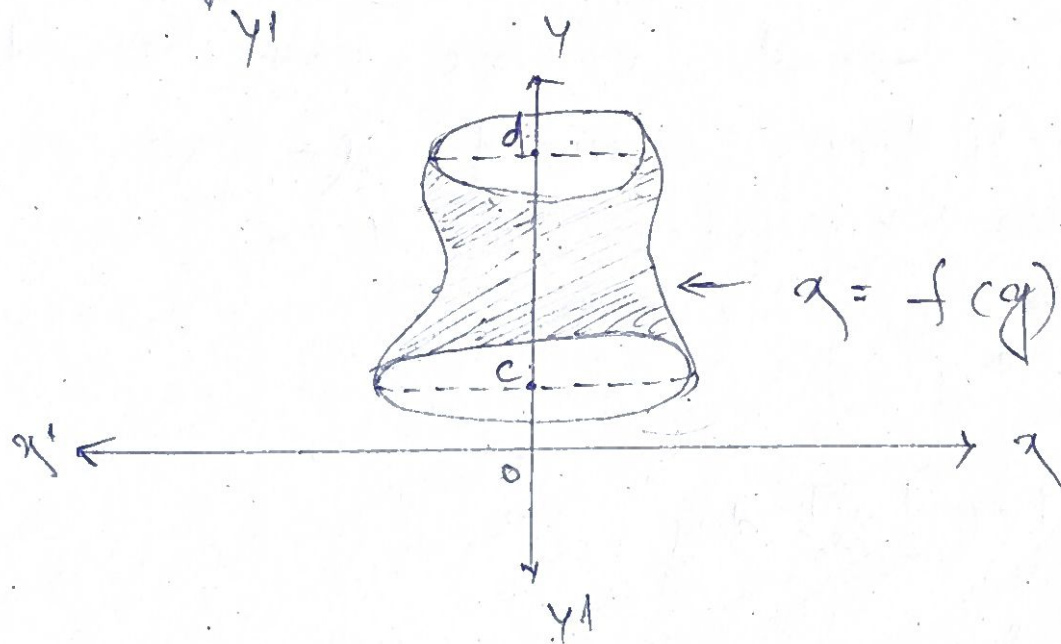
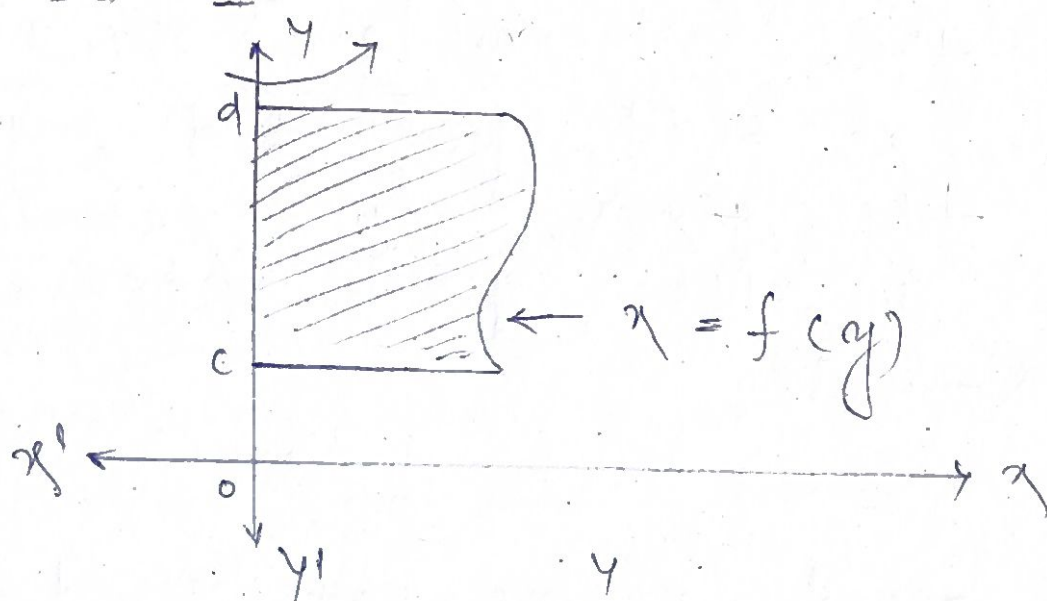
$x$  - axis  $Ox$ ,

$$V = \pi \int_a^b (f(x))^2 dx$$

$$V = \pi \int_a^b y^2 dx$$

$$V = \int_a^b \pi y^2 dx$$

Case - II



The volume of a solid formed by revolution of an area about  $y$ -axis bounded by the curve  $x = f(y)$ , the lines  $y = c$ ,  $y = d$  and the  $y$ -axis is,

$$V = \int_c^d \pi x^2 dy$$



$$\Rightarrow V = \pi \int_c^d (f(y))^2 dy$$

$$\Rightarrow V = \pi \int_c^d y^2 dy$$

$$\Rightarrow V = \int_c^d \pi y^2 dy$$

Example-20:-

Let  $y = f(x) = x^2$  be the curve given on the interval  $[0, 2]$ . Find the volume of solid generated by revolving the given region between the curve in the given interval around  $x$ -axis.

Solution:-

We know that volume of solid of revolution around  $x$ -axis on  $[a, b]$  is given by (for  $y = f(x)$ )

$$V = \int_a^b \pi y^2 dx$$

$$\Rightarrow V = \int_0^2 \pi [x^2]^2 dx$$

$$\Rightarrow V = \int_0^2 \pi x^4 dx$$

$$\Rightarrow V = \pi \int_0^2 x^4 dx$$

$$\Rightarrow V = \pi \left[ \frac{x^{4+1}}{4+1} \right]_0^2$$

$$\Rightarrow V = \pi \left[ \frac{x^5}{5} \right]_0^2$$

$$\Rightarrow V =$$

$$\Rightarrow V =$$

$$\Rightarrow V =$$

$$\Rightarrow V =$$

$$\Rightarrow V =$$

Example

Let the volume of solid generated by revolving the curve around

Solution:-

The volume of solid of revolution is given by

$$V =$$

$$=$$

$$\Rightarrow V =$$

$$\Rightarrow V =$$

$$\Rightarrow V =$$

$$\Rightarrow V =$$



$$\Rightarrow V = \pi \left[ \frac{x^5}{5} \right]_0^2$$

$$\Rightarrow V = \pi/5 [2^5 - 0^5]$$

$$\Rightarrow V = \pi/5 [32 - 0]$$

$$\Rightarrow V = \pi/5 \times 32$$

$$\Rightarrow V = \frac{32\pi}{5}$$

Example :- 2

Let  $y = f(x) = x^2$  be given on the interval  $[0, 2]$ . find the volume of solid generate by revolving the region between the curve (in the given interval  $[0, 2]$ ) around  $y$ -axis.

Solution :-

We have to find the revolving around  $y$ -axis in  $[a, b]$  is given by  $a=0$ ,  $b=2$ .

$$V = \int_a^b [f(y)]^2 dy$$

$$\Rightarrow V = \int_0^2 \pi x^2 dy$$

$$\Rightarrow V = \pi \int_0^2 y dy$$

$$\Rightarrow V = \pi \left[ \frac{y^{1+1}}{1+1} \right]_0^2$$

$$\Rightarrow V = \pi \left[ \frac{y^2}{2} \right]_0^2$$



$$\Rightarrow V = \pi/2 [y^2]_0^2$$

$$\Rightarrow V = \pi/2 [2^2 - 0^2]$$

$$\Rightarrow V = \pi/2 [4 - 0]$$

$$\Rightarrow V = \pi/2 \times 4$$

$$\Rightarrow V = 4\pi/2$$

$$\Rightarrow V = 2\pi$$

Example - 22

Evaluate the volume of the solid generated by the revolution of an ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

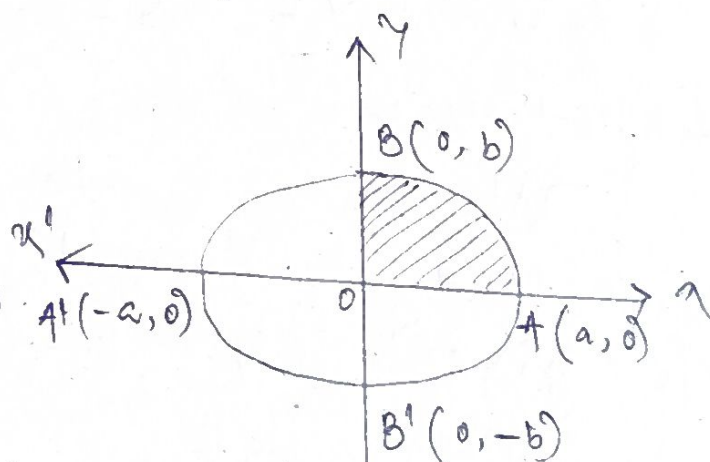
about  $Ox$ 's minor axis

Solution:-

Minor axis of the

ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ; ( $b < a$ )

is the  $y$ -axis



$\therefore$  We use the formula for revolution around  $y$ -axis i.e.

$$V = \int_a^b \pi x^2 dy \text{ on interval } [a, b]$$

Now, equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow \frac{x^2}{a^2} = 1 - \frac{y^2}{b^2}$$



$$\Rightarrow \frac{y^2}{a^2} = \frac{b^2 - y^2}{b^2}$$

$$\Rightarrow y^2 = a^2 \left( \frac{b^2 - y^2}{b^2} \right)$$

$\therefore$  Volume of the ellipse about its minor axis ( $y$ -axis).

= Volume generated by revolution of arc  $B'A$  about  $y$ -axis

= Twice the volume generated by revolution of the arc of  $BA$  about  $y$ -axis (as ellipse is symmetrical about  $x$ -axis). But the arc  $BA$  varies from  $y=0$  to  $y=b$

$$V = 2 \int_a^b \pi y^2 dy$$

$$\Rightarrow V = 2 \int_0^b a^2 \left( \frac{b^2 - y^2}{b^2} \right) dy$$

$$\Rightarrow V = 2 \int_0^b \pi \frac{a^2}{b^2} (b^2 - y^2) dy$$

$$\Rightarrow V = \frac{2\pi a^2}{b^2} \int_0^b (b^2 - y^2) dy$$

$$\Rightarrow V = \frac{2\pi a^2}{b^2} \left[ \int_0^b b^2 dy - \int_0^b y^2 dy \right]$$

$$\Rightarrow V = \frac{2\pi a^2}{b^2} \left[ b^2 \int_0^b dy - \left[ \frac{y^{2+1}}{2+1} \right]_0^b \right]$$

$$\Rightarrow V = \frac{2\pi a^2}{b^2} \left[ b^2 [y]_0^b - \left[ \frac{y^3}{3} \right]_0^b \right]$$



$$\Rightarrow V = \frac{2\pi a^2}{b^2} \left[ b^2 [b - 0] - \left[ \frac{b^3}{3} - \frac{0^3}{3} \right] \right]$$

$$\Rightarrow V = \frac{2\pi a^2}{b^2} \left[ b^2 \times b - \frac{b^3}{3} \right]$$

$$\Rightarrow V = \frac{2\pi a^2}{b^2} \left( b^3 - \frac{b^3}{3} \right)$$

$$\Rightarrow V = \frac{2\pi a^2}{\cancel{b^2}} \times \frac{\cancel{2}b^3}{3}$$

$$\Rightarrow V = \frac{2\pi a^2 \times 2}{3}$$

$$\Rightarrow V = \frac{4\pi a^2}{3}$$



$$\Rightarrow V = \frac{2\pi a^2}{b^2} \left[ b^2 \times b - \frac{b^3}{3} \right]$$

$$\Rightarrow V = \frac{2\pi a^2}{b^2} \left( b^3 - \frac{b^3}{3} \right)$$

$$\Rightarrow V = \frac{2\pi a^2}{b^2} \times \frac{2b^3}{3}$$

$$\Rightarrow V = \frac{2\pi a^2 \times 2}{3}$$

$$\Rightarrow V = \frac{4\pi a^2}{3}$$

Unit-03

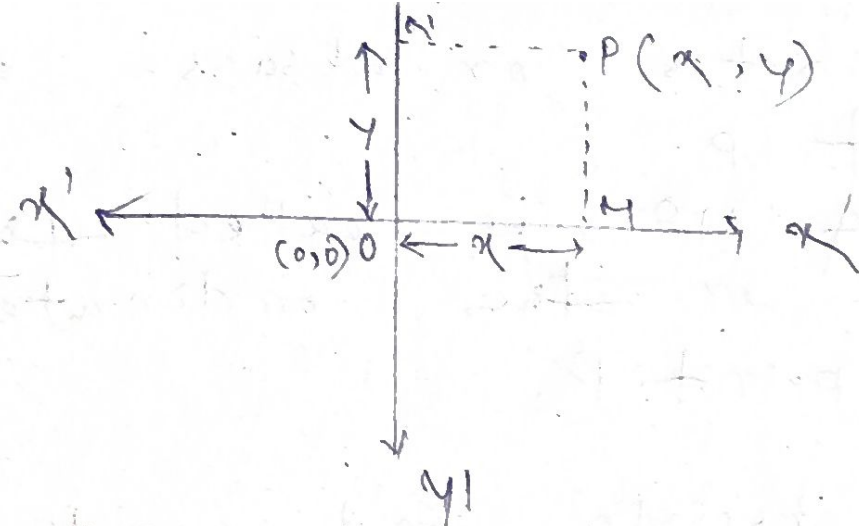
Co-ordinate Geometry

Definition of co-ordinate Geometry:  
It is a branch of Geometry which set-up a definite correspondence between the position of point in a plane & a pair of algebraic numbers called ordinates.

Cartesian co-ordinates :-

\* The number times distances from a fixed point are marked in equal units positively in one direction & negatively in the other.





- ⇒ To perpendicular number lines are one horizontal & other vertical.
- ⇒ Those two number lines are called  $xx'$  &  $yy'$ .
- ⇒ These two number lines cross each other at  $O$  also called origin.
- ⇒ The horizontal line  $xx'$  is called  $x$ -axis & vertical line  $yy'$  is called  $y$ -axis.
- ⇒ In cartesian co-ordinate the position of a point  $P$  is determined by knowing the distance from the two perpendicular line passing through the fixed point.
- ⇒ Let  $xx'$  &  $yy'$  are two perpendicular lines through origin is called cartesian or regular co-ordinate axis.
- ⇒  $PM$  and  $PN$  are perpendicular on  $ox$  &  $oy$  respectively.



- OM & MP is called the  $x$ -coordinates or abscissa of the point P.
- ON & NP is called the  $y$ -coordinates or the ordinates of the point P.
- The abscissa and ordinates of the point taken together is known as co-ordinates of a point.
- If  $OM = x$ ,  $ON = y$  the coordinates of the point P are  $(x, y)$  these coordinates are called the cartesian coordinates of P.

## Quadrant

→ The axis divided the plane into 4 parts. Those 4 parts are called quadrant.

→ Q. The plane consist of axis & coordinates the plane is called  $x'y'$  the cartesian plane or the  $xy$  plane or coordinate plane these are called the coordinate axis.

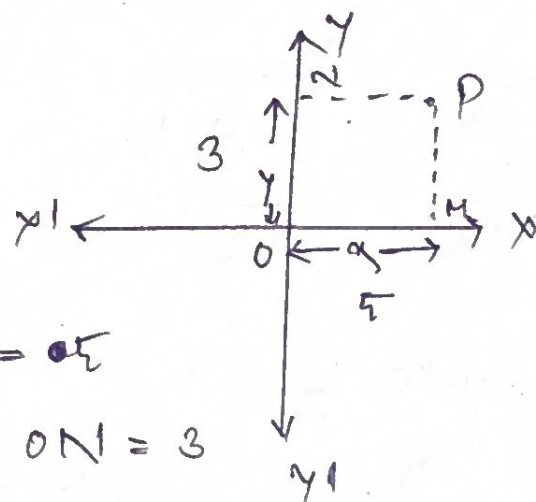
- i.  $xoy$  - I quadrant  $\rightarrow (+, +)$
- $x'o'y$  - II quadrant  $\rightarrow (-, +)$
- $x'oy'$  - III quadrant  $\rightarrow (-, -)$



xy - IV quadrant  $\rightarrow (+, -)$

Que:-

Solution



1. abscissa  $PM$  or  $OM = 5$

2. Ordinates  $PM$  or  $ON = 3$

3. Coordinates of point  $P = (x, y)$   
 $= (5, 3)$

State line:-

Consider a two points on geometrical space  $P$  &  $Q$ . The shortest distance between the two points  $P$  &  $Q$  is called state line.

If  $P$  &  $Q$  are any two points denoted as  $P(x_1, y_1)$  &  $Q(x_2, y_2)$  then distance between  $P$  &  $Q$  is,

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example:-

$$P = (3, 4)$$

$$Q = (4, 5)$$

Solution:-

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\Rightarrow PQ = \sqrt{(4 - 3)^2 + (5 - 4)^2}$$

$$x_1 = 3, y_1 = 4$$

$$x_2 = 4, y_2 = 5$$



$$\Rightarrow PQ = \sqrt{(1)^2 + (1)^2}$$

$$\Rightarrow PQ = \sqrt{1+1}$$

$$\Rightarrow PQ = \sqrt{2}$$

Example :-

The number of points on x-axis, which are at a distance 'a' from another point Q(3, 4) find the values of a.

Solution :-

$$x_1 = 3 \quad x_2 = x_1$$

$$y_1 = 4 \quad y_2 = 0$$

$$a = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\Rightarrow a = \sqrt{(x_1 - 3)^2 + (0 - 4)^2}$$

$$\Rightarrow a^2 = (x_1 - 3)^2 + (0 - 4)^2$$

$$\Rightarrow a^2 = (x_1 - 3)^2 + 16$$

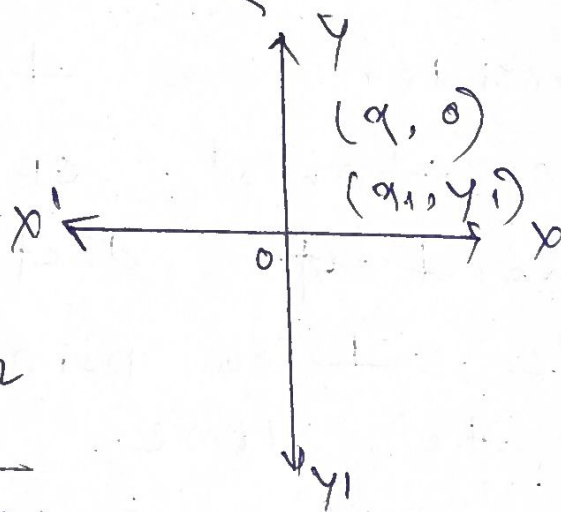
$$\Rightarrow a^2 = (x_1 - 3)^2 + 16$$

$$\Rightarrow a^2 - 16 = (x_1 - 3)^2$$

$$\Rightarrow \pm \sqrt{a^2 - 16} = x_1 - 3$$

$$\Rightarrow \pm \sqrt{a^2 - 16} + 3 = x_1$$

$$\Rightarrow x_1 = 3 \pm \sqrt{a^2 - 16}$$



Equation of line  
1. Equation of line  
is  $x = 0$

2. Equation of line  
parallel to  
at distance  
units is  
or  $x = -$

Equation of line  
1. Equation of line  
is  $y = 0$

2. Equation of line  
parallel to  
at a distance  
'b' units  
 $y = -b$

Note :-

→ Relation  
satisfied  
every point  
as equation  
it is given

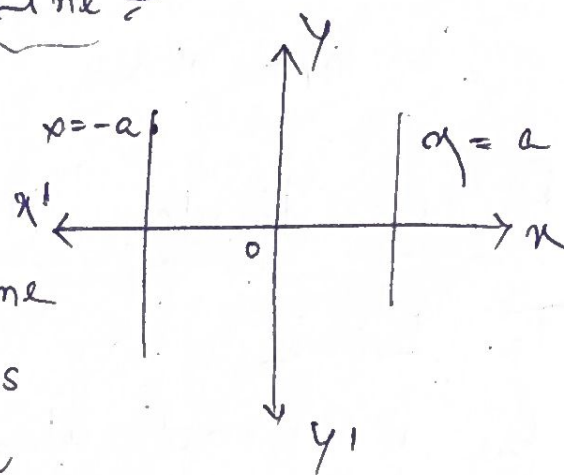
Parallel lines  
Two lines can  
be parallel  
not have



## Equation of vertical line :-

1. Equation of  $y$ -axis  
is  $x = 0$

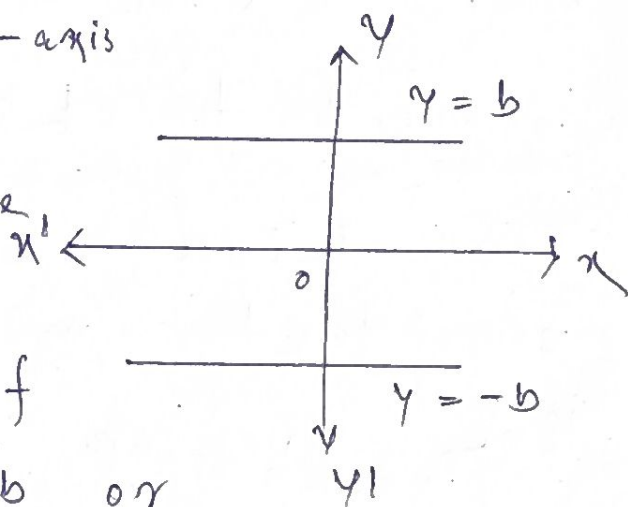
2. Equation of a line  
parallel to  $y$ -axis  
at distance of  $a$   
units is  $x = a$   
or  $x = -a$



## Equation of Horizontal Line :-

1. Equation of  $x$ -axis  
is  $y = 0$

2. Equation of a line  
parallel to  $x$ -axis  
at a distance of  
' $b$ ' units is  $y = b$  or  
 $y = -b$

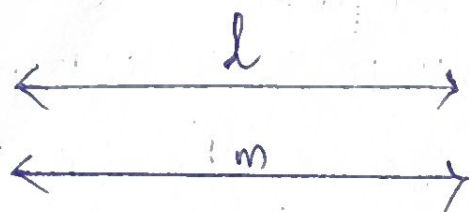


## Note :-

→ Relation in  $x$  &  $y$  which is  
satisfied by coordinates of  
every point on a line is known  
as equation of the line  
if it is given by  $ax + by + c = 0$   
( $\therefore a, b \neq 0$ )

## Parallel Line :-

Two lines are said to  
be parallel if they do  
not have any intersection point this  
implies that the two lines are disjoint



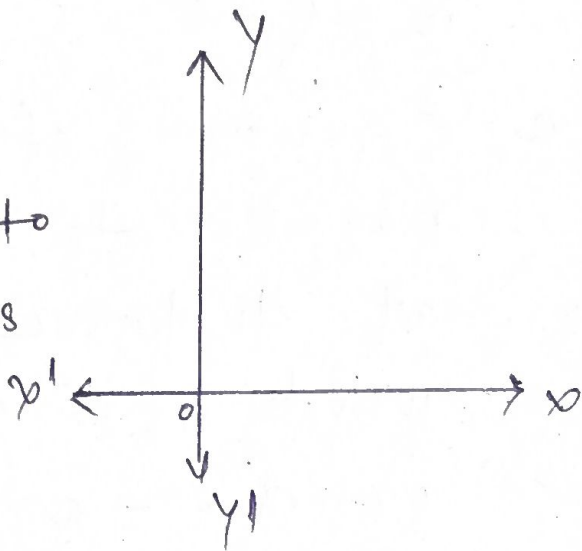


→ They are identical line

$$m_1 = m_2$$

Perpendicular lines:

Two lines are said to be perpendicular lines if they intersect at  $90^\circ$  angle.



→ Their slopes are negative reciprocal  
i.e.  $m_1 \times m_2 = -1$

Coincident lines:

Two lines are said to be coincident on each other if one line overlaps the other line.

→ Two parallel lines moving in the same direction they will meet at a point which is called point at infinity but geometrically it is not possible.

Slope of a line:

Suppose a line PQ makes an angle  $\phi$  ( $0^\circ \leq \phi \leq 180^\circ$ ) with the right side direction of x-axis, then slope of this line will be designated by  $\tan \phi$  & is denoted by symbol  $m$  i.e.  $m = \tan \phi$

If  $P(x_1, y_1)$  &  $Q(x_2, y_2)$  are two

points  
slope

$$PQ =$$

or,

Slope of

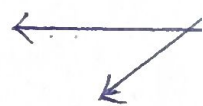
$$PQ = \frac{\text{Change in } y}{\text{Change in } x}$$

$$= \frac{\Delta y}{\Delta x}$$

$$= \frac{y_2 - y_1}{x_2 - x_1}$$

Positive

The line rises



Negative

The line falls  
right. -ve





points such that  $x_1 \neq x_2$ , then slope of the line.

$$PQ = \frac{y_2 - y_1}{x_2 - x_1}$$

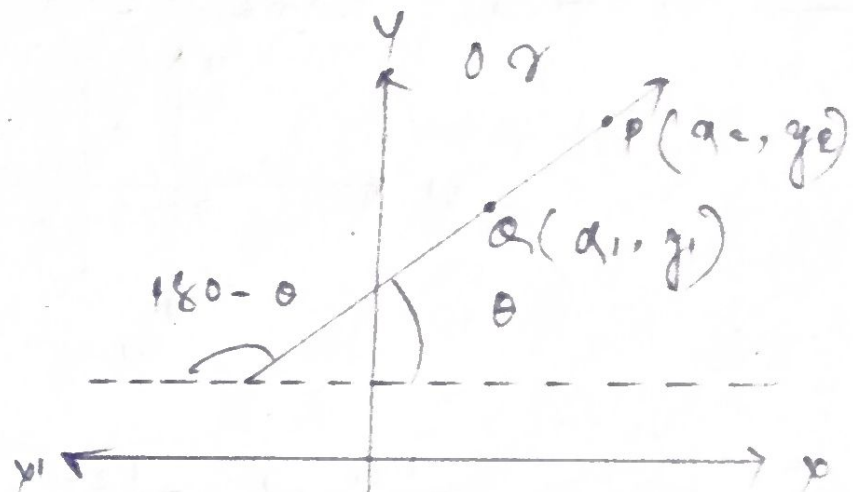
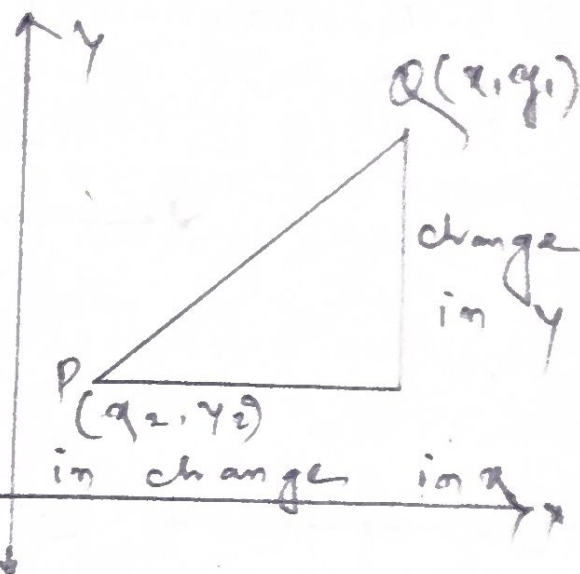
or,

Slope of the line

$$PQ = \frac{\text{Change in } y}{\text{Change in } x}$$

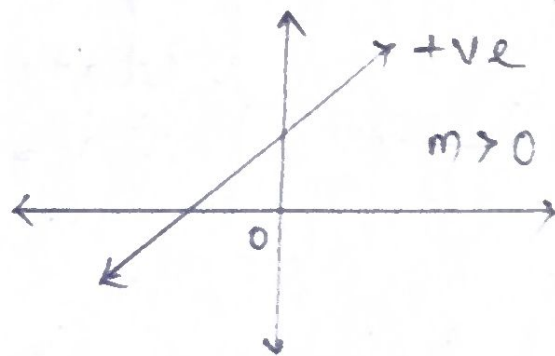
$$= \frac{\Delta y}{\Delta x}$$

$$= \frac{y_2 - y_1}{x_2 - x_1}$$



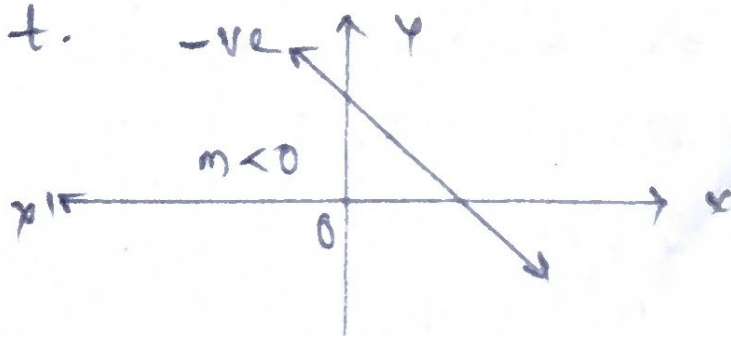
Positive Slope ( $m > 0$ )

The line rises from Right



Negative Slope ( $m < 0$ )

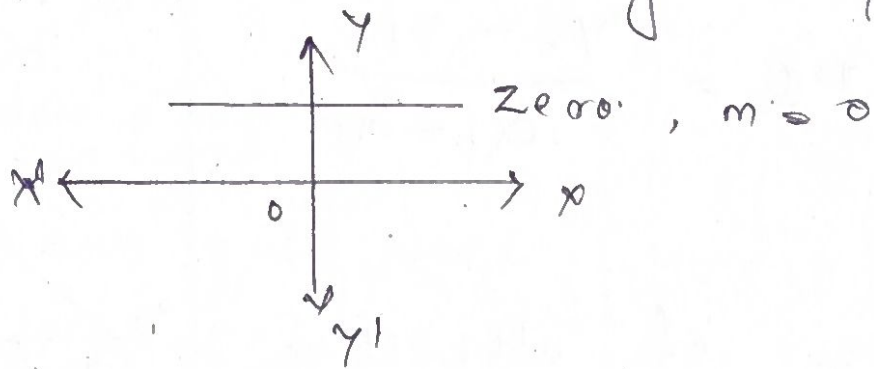
The Line falls from left to right.





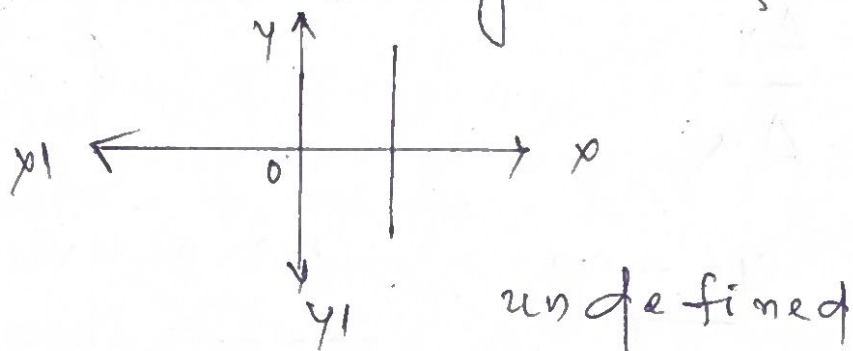
Zero Slope ( $m=0$ )

A horizontal line (change in  $y$ -axis is 0)



Undefined Line:

A vertical line (change in  $x$  is 0)



(b)

Q. Prove that the lines

$$(a) \quad 2x + 5y + c, \quad 4x + 10y + 2c = 0$$

are parallel to each other

Solution:

Let,

$$2x + 5y + c = 0 \quad \text{--- i}$$

equation of  
state line  
 $\therefore y = mx + c$

$$\Rightarrow 5y = -2x - c$$

$$\Rightarrow y = \frac{-2x - c}{5}$$

$$\Rightarrow y = -\frac{2x}{5} - \frac{c}{5} \quad m_1 = -\frac{2}{5}$$

$$4x + 10y + 2c = 0 \quad \text{--- ii}$$

$$\Rightarrow 10y = -4x - 2c$$

$$\Rightarrow y = \frac{-4x - 2c}{10}$$



$$\Rightarrow y = -\frac{1}{10}x - \frac{201}{10}$$

$$\Rightarrow y = -\frac{2}{5}x - \frac{201}{10} \quad m_2 = -\frac{2}{5}$$

$$-\frac{2}{5} = -\frac{2}{5}$$

$\therefore m_1 = m_2$   $\rightarrow$  two linear eqn<sup>s</sup> are parallel.

$$(b) 3x + 4y + 7 = 0$$

$4x - 3y + 1 = 0$  are perpendicular to each other.

Solution :-

$$3x + 4y + 7 = 0 \quad \text{--- i}$$

$$\Rightarrow 4y = -3x - 7$$

$$\Rightarrow y = \frac{-3x - 7}{4}$$

$$\Rightarrow y = -\frac{3}{4}x - \frac{7}{4} \quad m_1 = -\frac{3}{4}$$

$$4x - 3y + 1 = 0 \quad \text{--- ii}$$

$$\Rightarrow 4x + 1 = 3y$$

$$\Rightarrow \frac{4x + 1}{3} = y$$

$$\Rightarrow \frac{4}{3}x + \frac{1}{3} = y \quad m_2 = \frac{4}{3}$$

$\therefore$  Perpendicular line formula

$$\text{is } m_1 \times m_2 = -1$$

$$= -\frac{3}{4} \times \frac{4}{3} = -1$$



Equation of straight line in various standard forms

i. Slope intercept form is

Suppose  $m$  be the slope of a line and  $c$  its intercept on  $y$ -axis. Then the equations of such straight line are given as

$$y = mx + c \quad \text{--- i}$$

where  $y$  = depended variable  
 $x$  = Independent variable

If in this equation you put  $c = 0$

$c$  value put in equation 1

$$y = mx + c$$

$$\Rightarrow y = mx + 0$$

$$\Rightarrow y = mx$$

Point Slope form :-

If small  $m$  be the slope of a straight line which passes through a point  $(x_1, y_1)$  then its equation is written as  $y - y_1 = m(x - x_1)$

Intercept form :-

Suppose  $c_1$  &  $c_2$  be the two intercepts obtained by a straight line on the  $x$ -axis &  $y$ -axis

then the  $\frac{x}{c_1}$

$$i. P = (c_1, 0)$$

$$Q = (0, c_2)$$

Solution :-

$$PQ = \sqrt{(x_2 - x_1)^2}$$

$$\Rightarrow PQ = \sqrt{(c_1 - 0)^2}$$

$$\Rightarrow PQ = \sqrt{c_1^2 +}$$

$$\Rightarrow PQ = \sqrt{c_1^2 +}$$

ii. Area of  $\triangle$

iii. Two point  
 Equation of two points (expressed as

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

where  $|x|$  den of  $x$



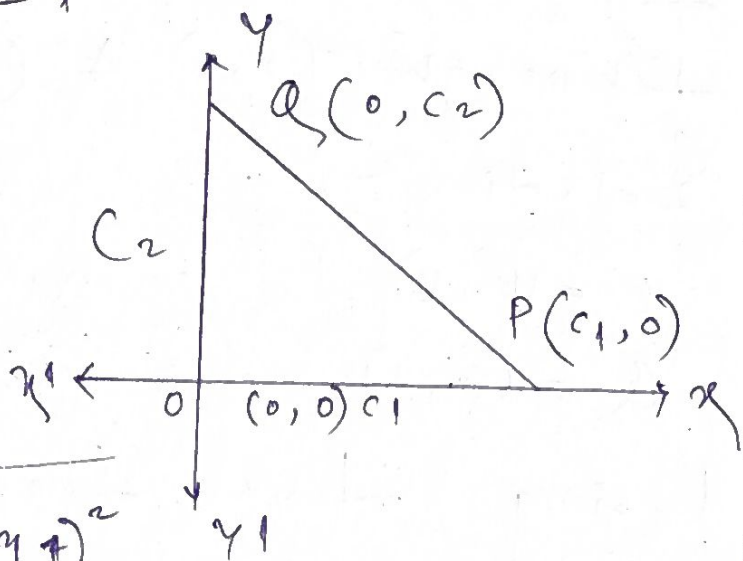
Then the eqn is written as

$$\frac{x}{c_1} + \frac{y}{c_2} = 1$$

i.  $P = (c_1, 0)$

$Q = (0, c_2)$

Solution :-



$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\Rightarrow PQ = \sqrt{(c_1 - 0)^2 + (0 - c_2)^2}$$

$$\Rightarrow PQ = \sqrt{c_1^2 + (-c_2)^2}$$

$$\Rightarrow PQ = \sqrt{c_1^2 + c_2^2}$$

ii. Area of triangle OPQ =  $\frac{1}{2} PO \cdot QO$

$$= \left| \frac{1}{2} c_1 \cdot c_2 \right|$$

iii. Two point form :-

Equation of a line going through two points  $(x_1, y_1)$  &  $(x_2, y_2)$  is expressed as

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1) \Leftrightarrow \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Where  $|x|$  denotes determinant of  $x$



Example-3 :-  
find the equation of line passing through  $(0,0)$  &  $(2,2)$

Solution :-

$$P = (0,0)$$

$$x_1 = 0$$

$$y_1 = 0$$

$$Q = (2,2)$$

$$x_2 = 2$$

$$y_2 = 2$$

Using the formula

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1)$$

$$\Rightarrow y - 0 = \frac{2 - 0}{2 - 0} (x - 0)$$

$$\Rightarrow y = \cancel{2/2} x$$

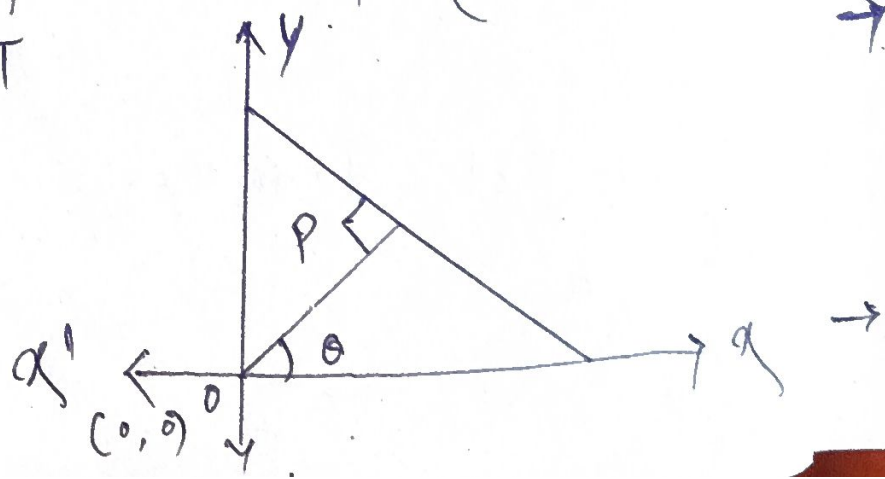
$$\Rightarrow y = x$$

Normal form :-

If  $p$  is the distance of perpendicular on a straight line from the origin, and  $\theta$  is the angle between the positive  $x$ -axis & this perpendicular, then equation of this is expressed as

$$x \cos \theta + y \sin \theta = p \quad (p > 0)$$

where  $0 \leq \theta \leq 2\pi$

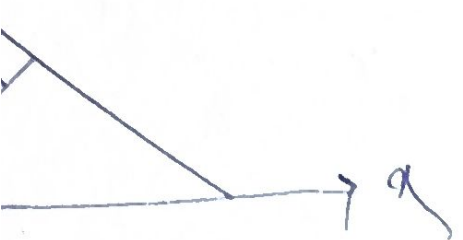




line passing

$= 0$   
 $= 2$

of  
straight line  
as the  
positive  
indicates,  
as expressed  
( $p > 0$ )



General form  
Consider two variables  $x$  &  $y$   
and three constants  $a, b, c$ . The  
linear or first degree equation  
in variable  $x$  &  $y$  is ~~perpendicular~~  
represented as

$$ax + by + c = 0 \quad \text{--- i}$$

- i. Slope of the line  $= \frac{-a}{b} = \frac{\text{Coefficient of } x}{\text{Coefficient of } y}$
- ii. Intercept by this line on  $x$ -axis  
 $= -\frac{c}{a}$  and also the intercept  
by this line on  $y$ -axis  $= -\frac{c}{b}$
- iii. In order to change the general  
form of straight line into the  
normal form we take  $c$  to the  
RHS of (i) and convert it to  
positive, then divide the entire  
equation by  $\sqrt{a^2 + b^2}$

### Slope

$$2x + 3y + 6 = 0$$

$$\Rightarrow m = \frac{-a}{b}$$

$$\Rightarrow m = -\frac{2}{3}$$

→ Example for  $x$  intercept

$$\Rightarrow x = -\frac{c}{a}$$

$$\Rightarrow x = -\frac{6}{2}$$

→ Example for  $y$  intercept

$$\Rightarrow y = -\frac{c}{b} = -\frac{6}{3}$$



Angle between two lines :-

a) Angle  $\phi$  between two straight lines (say)

$$y = a_1 x + c_1$$

$$y = a_2 x + c_2 \text{ is}$$

$$\tan \phi = \pm \left( \frac{a_1 - a_2}{1 + a_1 a_2} \right)$$

Note :-

There are two angle generated between two straight lines but usually the acute angle is taken as the angle between two lines hence the value of  $\phi$  can be determined on considering only the positive value of  $\tan \phi$ .

$$\tan \phi = \frac{a_1 - a_2}{1 + a_1 a_2}$$

ii. Let  $a_1, a_2, a_3$  be the slopes of three straight lines

$$L_1 = 0; L_2 = 0; L_3 = 0, \text{ where}$$

$a_1 > a_2 > a_3$  then the interior angle of any straight line found by these formulas are given by  
( $\triangle EFG$ )

$$\tan E = \frac{a_1 - a_2}{1 + a_1 a_2}, \quad \tan F = \frac{a_1 - a_3}{1 + a_1 a_3}$$

$$\tan G = \frac{a_3 - a_1}{1 + a_3 a_1}$$

b) Consider lines

$a_1 x$

and

then +

i. Para

ii. Perp

iii. Coi

Remark :-

Two straight lines intersect at a point and form the straight line equation

$$y = a_1 x$$

Example is

Which value  $a_1$

is  $a_1$  and  $a_2$  of two



b) Consider the equation of two straight lines

$$a_1x + b_1y + c_1 = 0$$

$$a_2x + b_2y + c_2 = 0$$

Then these two straight lines are

i. Parallel  $\Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

ii. Perpendicular  $\Leftrightarrow a_1a_2 + b_1b_2 = 0$

$$(\because a_1a_2 = -1)$$

iii. Coincident  $\Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

$$(\because \Leftrightarrow = \text{if and only if})$$

Remark :-

A straight line making a given angle with a line equations of the straight lines going through a point whose coordinates are  $(x_1, y_1)$  and forming an angle (say)  $\alpha$ , with the straight line whose linear equation is  $y = mx + c$  is written as

$$y - y_1 = \frac{m \pm \tan \alpha}{1 \mp m \tan \alpha} (x - x_1)$$

Example :-

Which of the following is correct value of  $3x + 4y - 5 = 0$  &  $2x + 4y + 6 = 0$  are two equations of two perpendicular line.



Solution :-

$$3x + 4y - 5 = 0 \quad \text{--- i}$$

$$\Rightarrow m_1 = -\frac{a}{b}$$

$$\Rightarrow m_1 = -\frac{3}{4}$$

$$2x + ky + 6 = 0 \quad \text{--- ii}$$

$$\Rightarrow m_2 = -\frac{a}{b}$$

$$\Rightarrow m_2 = -\frac{2}{k}$$

Perpendicular line formula Qs

$$m_1 \times m_2 = -1$$

$$\Rightarrow -\frac{3}{4} \times -\frac{2}{k} = -1$$

$$\Rightarrow \frac{6}{4k} = -1$$

$$\Rightarrow \frac{6}{4k} = -1 \Rightarrow 6 = -4k$$

$$\Rightarrow \frac{6}{-4} = -k \Rightarrow -\frac{3}{2} = -k$$

$$\Rightarrow -\frac{3}{2} = k$$

Distance of perpendicular from a point on a line:

Let us consider a straight line whose equation is given as.

$ax + by + c = 0$ , then, distance of perpendicular from a point  $P(x_1, y_1)$  is expressed as

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

In particular, perpendicular lying on the line  $ax + by + c = 0$ .

Example :- 6 -

Verify whether the line drawn through  $R(1, -6)$  and  $P(-2, 3)$  is perpendicular to the line  $3x + 4y - 5 = 0$ .

Solution :-

Let the straight line passing through  $R(1, -6)$  and  $P(-2, 3)$  be

$$R = (1, -6) \text{ and } P = (-2, 3)$$

is

We know the equation of a line passing through two points  $(x_1, y_1)$  and  $(x_2, y_2)$  is

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow y - (-6) = m(x - 1)$$

$$\Rightarrow y + 6 = m(x - 1)$$

$$\Rightarrow y = m(x - 1) - 6$$

$$\Rightarrow y = mx - m - 6$$

$$\Rightarrow mx - m - 6 = 0$$

Then, the distance of the line from a point



In particular, the distance of the perpendicular from the origin  $(0,0)$  lying on the line

$$ax + by + c = 0 \text{ is } p = \frac{|c|}{\sqrt{a^2 + b^2}}, \text{ (since } x=0, y=0 \text{)}$$

Example :- 6 -

Verify whether any line can be drawn through the point  $(1, -6)$  so that its distance from  $(-2, 3)$  will be equal to 10.

Solution :-

Let the equation of the straight line through a point  $R = (1, -6)$  & its distance from  $P = (-2, 3)$  is equal to 10.

We know that the points slope from.

$$y - y_1 = m(x - x_1) \quad \{ \because m = \text{slope} \}$$

$$\Rightarrow y - (-6) = m(x - 1)$$

$$\Rightarrow y + 6 = m(x - 1)$$

$$\Rightarrow y = m(x - 1) - 6$$

$$\Rightarrow y = mx - m - 6$$

$$\Rightarrow mx - m - 6 - y = 0$$

Then, the distance of the perpendicular from a point  $(-2, 3)$  is equal to 10.



$$P = \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|$$

$$\Rightarrow \frac{mx - y - 4m - 6}{\sqrt{(m)^2 + (-1)^2}} = 10$$

$$\Rightarrow \frac{|m \times (-2) - 3 - 4m - 6|}{\sqrt{m^2 + 1}} = 10$$

$$\Rightarrow \frac{|-2m - 9 - 4m|}{\sqrt{m^2 + 1}} = 10$$

$$\Rightarrow \frac{|-6m - 9|}{\sqrt{m^2 + 1}} = 10$$

$$\Rightarrow \frac{|-(6m + 9)|}{\sqrt{m^2 + 1}} = 10$$

$$\Rightarrow \frac{6m + 9}{\sqrt{m^2 + 1}} = 10 \Rightarrow 6m + 9 = 10\sqrt{m^2 + 1}$$

1. Squaring Both side

$$\Rightarrow (6m + 9)^2 = (10\sqrt{m^2 + 1})^2$$

$$\Rightarrow 36m^2 + 81 + 2 \cdot 6m \cdot 9 = 100(m^2 + 1)$$

$$\Rightarrow 36m^2 + 81 + 108m = 100m^2 + 100$$

$$\Rightarrow 100m^2 - 36m^2 - 108m + 100 - 81 = 0$$

$$\Rightarrow 64m^2 - 108m + 19 = 0$$

Now, the discriminant  $b^2 - 4ac$  is

$$\Rightarrow (-108)^2 -$$

$$\Rightarrow 11664 -$$

$$\Rightarrow 6800 = 0$$

$$\therefore b^2 - 4ac$$

Distance, Distance

Distance

Distance between

Distance between lines.

$$ax + by + c$$

$$ax + by + c$$

$$D = \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$$

In this case the coefficient of equations are not

The Area of

Line  $Q_1$  &  $Q_2$   
Two pairs of

$\theta$  is the angle adjacent edges

Area of the figure by the four



$$\Rightarrow (-108)^2 - 4 \times 64 \times 19 > 0$$

$$\Rightarrow 11664 - 4864 = 0$$

$$\Rightarrow 6800 = 0 \quad (6800 > 0)$$

$$\therefore b^2 - 4ac > 0$$

Hence, the equation of the straight line is possible.

Distance between two parallel lines

Distance between two disjoint lines.

$$ax + by + c_1 = 0 \text{ and}$$

$$ax + by + c_2 = 0 \text{ is determined as}$$

$$D = \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$$

In this case parallel disjoint lines are coefficient of  $x$  and  $y$  in both equations must be same.

$$\rightarrow \text{The Area of parallelogram} = \frac{Q_1 \times Q_2}{\sin \theta}$$

Where  $Q_1$  &  $Q_2$  are length between two pairs of opposite edge &  $\theta$  is the angle between any two adjacent edges.

Area of the parallelogram bounded by the four straight lines say.



$$y = m_1 x + c_1 \text{ --- i}$$

$$y = m_2 x + d_1 \text{ --- ii}$$

$$y = m_1 x + c_2 \text{ --- iii}$$

$$y = m_2 x + d_2 \text{ --- iv}$$

Then,

$$z = \left| \frac{(c_1 - c_2)(d_1 - d_2)}{m_1 - m_2} \right|$$

Example :-

The given equations of 3 lines are.

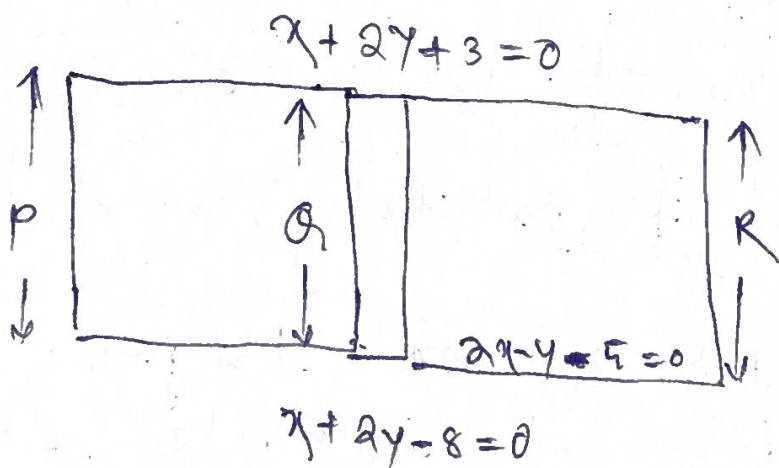
$$x + 2y + 3 = 0 \text{ --- i}$$

$$x + 2y - 8 = 0 \text{ --- ii}$$

$$2x - y - 5 = 0 \text{ --- iii}$$

form 3 sides of two squares. find the equations of remaining sides of these squares.

Solution :-



Use of the formula to find the distance between the two parallel lines as

$$x + 2y + 3 = 0 \text{ --- i}$$

$$x + 2y$$

$$a = 1, b =$$

$$D = \frac{|c_1 - c_2|}{\sqrt{a^2}}$$

$$\Rightarrow D = \frac{|3 - (-8)|}{\sqrt{(1)^2}}$$

$$\Rightarrow D = \frac{|3 + 8|}{\sqrt{1 + 4}}$$

$$\Rightarrow D = \frac{|11|}{\sqrt{5}} =$$

The equation of the lines are of the form

$$2x - y$$

$$2x - y$$

Distance between the two parallel lines

$$\frac{|c_1 - c_2|}{\sqrt{a^2}}$$

$$c_1 = 3, c_2 = -8$$

$$a = 1, b = 2$$

$$\Rightarrow \frac{|3 - (-8)|}{\sqrt{(2)^2 + (-1)^2}}$$

$$\Rightarrow \frac{|3 + 8|}{\sqrt{4 + 1}} =$$

$$\Rightarrow \frac{|11|}{\sqrt{5}} =$$



$$x + 2y - 8 = 0 \text{ --- i}$$

$$a = 1, b = 2, c_1 = 3, c_2 = -8$$

$$D = \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$$

$$\Rightarrow D = \frac{|3 - (-8)|}{\sqrt{(1)^2 + (2)^2}}$$

$$\Rightarrow D = \frac{|3 + 8|}{\sqrt{1 + 4}}$$

$$\Rightarrow D = \frac{11}{\sqrt{5}} = \frac{11}{\sqrt{5}}$$

The equation of the sides P & R are of the form

$$2x - y + k = 0 \text{ --- i}$$

$$2x - y - 5 = 0 \text{ --- ii}$$

Distance between sides P & R = distance between sides Q and R

$$\frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}} = D$$

$$c_1 = k, c_2 = -5$$

$$a = 2, b = -1$$

$$\Rightarrow \frac{|k - (-5)|}{\sqrt{(2)^2 + (-1)^2}} = \frac{11}{\sqrt{5}}$$

$$\Rightarrow \frac{|k + 5|}{\sqrt{4 + 1}} = \frac{11}{\sqrt{5}}$$

$$\Rightarrow \frac{|k + 5|}{\sqrt{5}} = \frac{11}{\sqrt{5}}$$



$$|K+5| = 11$$

$$K+5 = +11$$

$$\Rightarrow K+5 = 11$$

$$\Rightarrow K = 11-5$$

$$\Rightarrow K = 6$$

$$\text{or } K+5 = -11$$

$$\Rightarrow K+5 = -11$$

$$\Rightarrow K = -11-5$$

$$\Rightarrow K = -16$$

The fourth equation of the sides ~~part~~ of the two squares are.

$$2x - y + K = 0$$

$$\Rightarrow 2x - y + 0 = 0$$

and

$$2x - y + K = 0$$

$$2x - y + 16 = 0$$

Concept of the circle :-

A circle is a closed two dimensional figure in which the set of the points in the plane is equidistant from a given fixed point called the 'Centre' of the circles and the constant distance is called the radius of the circle.

The Line joining any two points on the circumference is called chord.

→ The chord passes through centre the circle  $O$ .

it is

Diameter  
Circumference  
Area

Where

General  
Mathematical  
general

$x^2$

Where  
are con

(-)

(- coefficient)

find radius

Note :-

the circle

If



it is called diameter where,

$$RS = \text{chord}$$

$$PQ = \text{Diameter}$$

$$O = \text{Centre}$$

Diameter of a circle  $PQ = 2r$   
Circumference of the circle  $(C) = 2\pi r$

Area of the circle is  $A = \pi r^2$

Where  $\pi$  is constant

$$\therefore \pi = \frac{220}{7}$$

$$\pi = 3.14$$

General equation of the circle  
Mathematically formula of the  
general circle equation as

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Where  $x, y$  are variables and  $g, f, c$  are constants

$$(-g, -f) = \text{centre}$$

$$\left( -\frac{\text{coefficient of } x}{2}, -\frac{\text{coefficient of } y}{2} \right)$$

$$\text{Find radius } r = \sqrt{g^2 + f^2 - c}$$

Note :-

If radius  $r$  is real then the circle is known as real circle.

If radius  $r = 0$  then the circle is a point / vertex circle,



If radius is imaginary  
then the circle is also imaginary

$$(imaginary = 1 + 2i)$$

$$(i = \sqrt{-1})$$

Example :- Ex :-

find the centre and radius  
of the circle.

$$x^2 + y^2 - 3x - 4y + 1 = 0$$

Solution :-

Given,

$$x^2 + y^2 - 3x - 4y + 1 = 0$$

$$\Rightarrow x^2 + y^2 - \frac{3}{2}x - \frac{4}{2}y + \frac{1}{2} = 0$$

$$\Rightarrow x^2 + y^2 - 3x - 4y + 1 = 0$$

$$\Rightarrow x^2 + y^2 - 2 \cdot \frac{3}{2}x - 2 \cdot 2y + 1 = 0$$

$$g = -3/2, f = -2, c = 1$$

$$\text{Centre} = (-g, -f)$$

$$= -(-3/2), -(-2)$$

$$= (3/2, 2)$$

$$\text{radius} = \sqrt{g^2 + f^2 - c}$$

$$= \sqrt{(-3/2)^2 + (-2)^2 - 1}$$

$$= \sqrt{9/4 + 4 - 1}$$

$$= \sqrt{9/4 + 3}$$



$$= \sqrt{\frac{9+12}{4}}$$

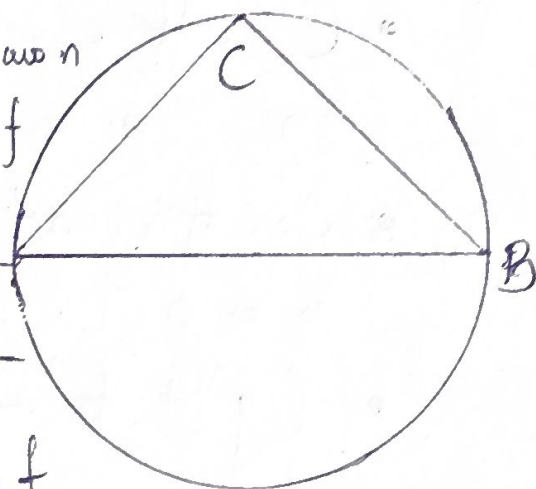
$$= \sqrt{\frac{21}{4}}$$

$$= \frac{\sqrt{21}}{2}$$

Characteristic of a circle :-

i. The diameter of a circle is the longest chord the circle and divides circle into two equal halves.

ii. The perpendicular drawn on any given chord of a circle from the centre of the circle bisect the chord (perpendicular bisector)



iii. For a given length of Diameter the circle is the shape with largest area.

iv. The circles are said to be congruent if they have equal radii.

v. The angle in a semi-circle is always  $90^\circ$ .

vi. All circles are similar regardless of the measure of radius or diameters.

vii. Equal chords of a circle subtend equal angles at the centre.

Note :-

1. The equation  $x^2 + y^2 + 2gx + 2fy + c = 0$  has three constants hence to get the general equation of the circle



at least three conditions should be known, this implies that a unique circle passes through three non collinear points.

Ex. of General second degree equation in  $x$  and  $y$  with variable  $h, g, f$  case constants is written as  $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$  or represents or ~~is written as~~ a circle if -

- Coefficient of  $x^2$  = coefficient of  $y^2$   
or  $a = b \neq 0$
- Coefficient of  $xy = 0$  implies that  $h = 0$
- $(g^2 + f^2 - c) \geq 0$  (for a real circle)

Equation of circle through three Given points :-

The equation of a circle given three points consider a three points

L =  $(x_1, y_1)$  M  $(x_2, y_2)$  and L  $(x_1, y_1)$

M  $(x_2, y_2)$

N  $(x_3, y_3)$

Let the equation of the required circle  $x^2 + y^2 + 2gx + 2fy + c = 0$

Since the three points  $(L, x_1, y_1)$ ,  $m(x_2, y_2)$  and  $N(x_3, y_3)$



full line on (i) so we have

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad \dots \text{eq (1)}$$

$$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0 \quad \dots \text{eq (2)}$$

$$x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0 \quad \dots \text{eq (3)}$$

Eliminating  $g, f$  and  $c$  from (i), (ii), (iii),  
(iv) we get

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

Corollary: If the four points  $(x_r, y_r)$   
where  $r = 1, 2, 3, 4$  lie on a circle then

$$\begin{vmatrix} x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \\ x_4^2 + y_4^2 & x_4 & y_4 & 1 \end{vmatrix} = 0$$

Example :-

find the equation of the circle which passes through the point  $L(0, 1)$ ,  $M(1, 0)$  and  $N(3, 2)$  also find the value of radius and coordinates of the center.

Solution :-

Suppose the general equation of a circle is given as.

$$x^2 + y^2 + 2gx + 2fy + c = 0$$



the point  $L(0, 1)$

$$(0)^2 + (1)^2 + 2g(0) + 2f(1) + c = 0$$

$$\Rightarrow 0 + 1 + 0 + 2f + c = 0$$

$$\Rightarrow 1 + 2f + c = 0 \quad \dots \dots \text{eq}^n (1)$$

the point  $M(1, 0)$

$$(1)^2 + (0)^2 + 2g(1) + 2f(0) + c = 0$$

$$\Rightarrow 1 + 0 + 2g + 0 + c = 0$$

$$\Rightarrow 1 + 2g + c = 0 \quad \dots \dots \text{eq}^n (2)$$

the point  $N(3, 2)$

$$(3)^2 + (2)^2 + 2g(3) + 2f(2) + c = 0$$

$$\Rightarrow 9 + 4 + 6g + 4f + c = 0$$

$$\Rightarrow 13 + 6g + 4f + c = 0 \quad \dots \dots \text{eq}^n (3)$$

from eq<sup>n</sup> (ii) & (iii)

we get

$$\begin{array}{r} 2f + c = -1 \\ 2g + c = -1 \\ \hline 2f - 2g = 0 \\ \Rightarrow 2(f - g) = 0 \\ \Rightarrow f - g = 0 \\ \Rightarrow f = g \end{array}$$

we get

eq<sup>n</sup> (iii)

$$\Rightarrow 13 + 6g + 4g + c = 0$$

$$\Rightarrow 13 + 10g + c = 0$$



$$2f(1) + c = 0$$

$$\dots \text{eq}^n (1)$$

$$(0) + c = 0$$

$$\dots \text{eq}^n (2)$$

$$(2) + c = 0$$

$\Rightarrow$

$$\dots \text{eq}^n (3)$$

we get

$$\text{eq}^n (iii)$$

$$\Rightarrow 13 + 6g + 4g + c = 0$$

$$\Rightarrow 13 + 10g + c = 0$$

Putting  $f$  value in eq<sup>n</sup> (iv)

$$13 + 6g + 4g + c = 0$$

$$10g + c = -13 \quad \text{--- (a)}$$

from eq<sup>n</sup> (a) and eq<sup>n</sup> (iii)

$$\text{we get, } 2g + c = 0$$

$$10g + c = 0$$

$$\begin{array}{r} - \\ \hline \end{array}$$

$$-8g = 12$$

$$\Rightarrow g = -\frac{12}{8}$$

$$\Rightarrow g = 1.5 \quad \text{or } f = -1.5$$

Putting  $g$  value in eq<sup>n</sup> (iii)

$$2(-1.5) + c = -1$$

$$\Rightarrow 3.0 + c = -1$$

$$\Rightarrow c = -1 + 3$$

$$\Rightarrow c = 2$$

$$\text{Center} = (-g, f)$$

$$= (1.5, 1.5)$$

$$\text{radius } r = \sqrt{g^2 + f^2 - c}$$

$$r = \sqrt{(1.5)^2 + (1.5)^2 - 2}$$

$$r = \sqrt{2.25 + 2.25 - 2}$$

$$r = \sqrt{2.50 - 2}$$

$$r = \sqrt{0.50}$$

$$r = \sqrt{0.5}$$

$$r = 0.707 \Rightarrow 0.7$$



The general equation  $x^2 + y^2 + 2gx + 2fy + c = 0$

$$\Rightarrow x^2 + y^2 + 2(-1.5)x + 2(-1.5)y + c = 0$$

$$\Rightarrow x^2 + y^2 + 3x - 3y + c = 0$$

Equation of circle in diameter form is

If  $A(x_1, y_1)$  and  $B(x_2, y_2)$  are the end points of the diameter of the circle and  $P(x, y)$  is the diameter of the circle

$\Rightarrow$  (Slope of PA) Slope of PB = -1 Then  $\angle APB = 90^\circ$

$$\Rightarrow \frac{y + y_1}{x - x_1} \times \frac{y - y_2}{x - x_2} = -1$$

$$\Rightarrow \frac{(y - y_1)(y - y_2)}{(x - x_1)(x - x_2)} = -1$$

$$\Rightarrow (y - y_1)(y - y_2) = -(x - x_1)(x - x_2)$$

$$\Rightarrow (y - y_1)(y - y_2) + (x - x_1)(x - x_2) = 0$$

Then  $\angle APB = 90^\circ$

$$P(x, y)$$

$$A(2, 3)$$

$$B(5, 6)$$

$$(y - y_1)(y - y_2) + (x - x_1)(x - x_2) = 0$$

$$\Rightarrow (y - 3)(y - 6) + (x - 2)(x - 5) = 0$$

$$\Rightarrow y^2 - 6y - 3y + 18 + x^2 - 5x + 10 = 0$$

$$\Rightarrow y^2 - 9y + 18 + x^2 - 5x - 2x + 10 = 0$$

$$\Rightarrow x^2 + y^2 - 7x - 9y + 28 = 0$$

## Unit-04 Vector Algebra

1. Introduction  
Scalar Quantities  
that have magnitude  
time, temperature  
Vector Quantities  
that have both  
magnitude and direction  
(e.g., velocity, acceleration)

2. Representation of a vector

A vector is represented by a line segment with arrows at both ends.

\* Initial point: The point where the vector starts.

\* Terminal point: The point where the vector ends.

\* Magnitude: The length of the vector, denoted by  $|V|$ .

Free vectors: Vectors that can be displaced parallel to themselves without changing their magnitude or direction are called free vectors.



$$1+c=0$$

$$\Rightarrow y^2 + y^2 - 7y = 4y + 28 = 0$$



# Vector Algebra

## Introduction to vectors

scalar quantities : Physical quantities that have only magnitude (e.g., mass, volume, time, temperature).

vector quantities : Physical quantities that have both magnitude and direction (e.g., velocity, force, acceleration).

Example 1 :-

Classify the following as scalars and vectors,

(a) 10 m North west

A :- It is distance with direction, so vector quantity.

(b)  $10^{-10}$  coulomb

A :- It is an electric charge, so scalar quantity.

(c) 20 Km/hr

A :- Speed is a scalar quantity.

(d) 15 m/s towards east

A :- distance with direction, vector quantity.

(e) 100 Newton

A :- It is a force, so vector quantity.

## Representation of vectors

A vector is denoted by a directed line segment

$\overrightarrow{AB}$  or a single letter  $\vec{a}$ .



## Free vectors :-

Vectors that can be displaced parallel to themselves without changing their magnitude or direction are called free vectors.

## Formula :-

$$l = \frac{x}{|\vec{r}|}, m = \frac{y}{|\vec{r}|}, n = \frac{z}{|\vec{r}|}$$

Key property :  $l^2 + m^2 + n^2 = 1$

## Direction ratios :-

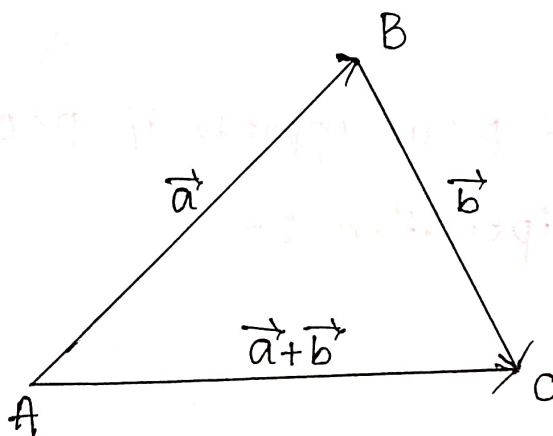
Any three numbers  $a, b, c$  proportional to the direction cosines  $l, m, n$  are called direction ratios.

$$\begin{matrix} l & m & n & = & k \\ a & b & c \end{matrix}$$

For a vector  $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ , the scalar components  $a, b, c$  are the direction ratios.

## Algebra of vectors

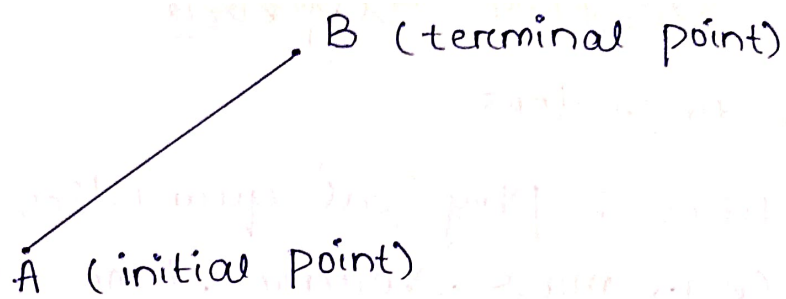
### • Triangle law :-



If two vectors are represented by two sides of a triangle in sequence (head to tail), their sum is the third side taken in the opposite order.

$$\boxed{\vec{AC} = \vec{AB} + \vec{BC}}$$





→ Initial point

The starting point (A) of the vector.

→ Terminal point

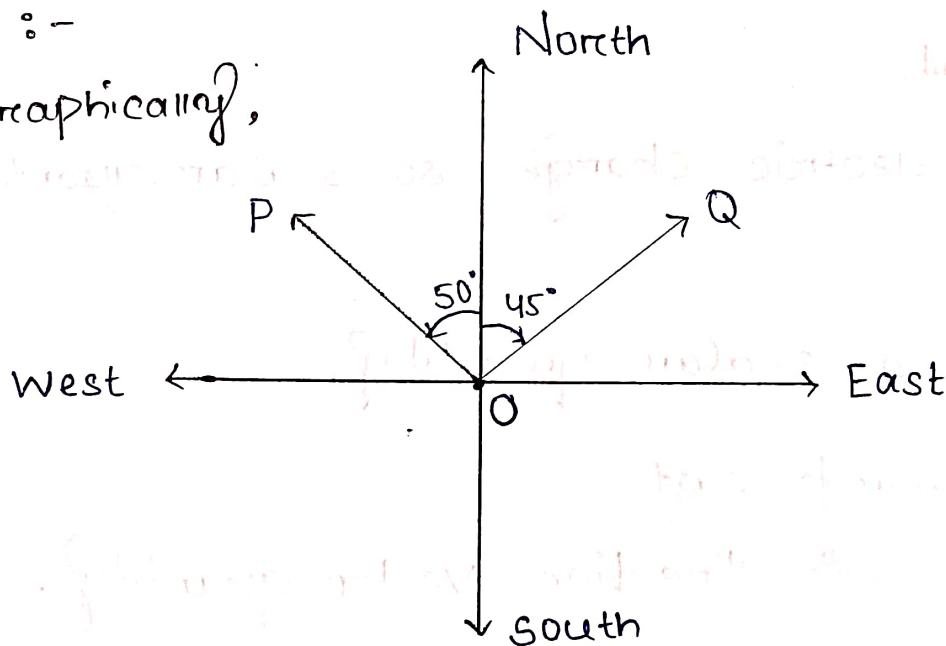
The ending point (B) of the vector.

→ Magnitude

The length of the vector, denoted by  $|\vec{AB}|$  or  $|\vec{a}|$  or  $a$ .

Example 2 :-

Represent graphically,



(i) A displacement of 80 km,  $50^\circ$  west of north

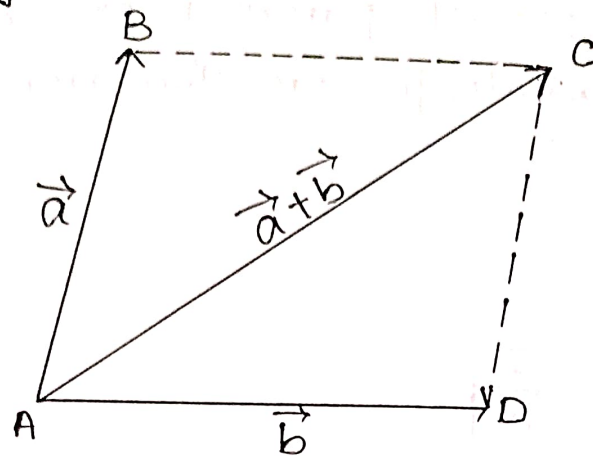
A :- The vector  $\vec{OP}$  represents required vector.

(ii) A displacement of 30 km North-east

A :- The vector  $\vec{OQ}$  represents required vector.

Note :- The magnitude i.e., distance b/w initial & terminal points of a vector is always a non-negative real number and is denoted as  $|\vec{OA}|$  or  $|\vec{a}|$  or  $a$ .





If two vectors originating from a point represent adjacent sides of a parallelogram, their sum is the diagonal originating from that same point.

### Properties of Vector Addition

- (1) Commutative :  $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
- (2) Associative :  $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$
- (3) Additive Identity :  $\vec{a} + \vec{0} = \vec{a}$
- (4) Additive Inverse :  $\vec{a} + (-\vec{a}) = \vec{0}$

Multiplication by a scalar : -

For a vector  $\vec{a}$  and scalars  $p, q$  :

- Magnitude :  $|p\vec{a}| = |p||\vec{a}|$
- Direction : Same as  $\vec{a}$  if  $p > 0$ , opposite if  $p < 0$ .

Properties of scalar multiplication : -

- (1)  $p(\vec{a} + \vec{b}) = p\vec{a} + p\vec{b}$
- (2)  $(p + q)\vec{a} = p\vec{a} + q\vec{a}$
- (3)  $p(q\vec{a}) = (pq)\vec{a}$



Example 3 :-

Evaluate  $a, b$  so that the vectors  $\vec{x} = a\hat{i} + 2\hat{j}$  and

$\vec{y} = 4\hat{i} + b\hat{j}$  are equal.

Sol<sup>n</sup> Given,

$$\vec{x} = a\hat{i} + 2\hat{j}$$

$$\vec{y} = 4\hat{i} + b\hat{j}$$

Two vectors are equal if their corresponding components are equal.

$$\vec{x} = \vec{y}$$

$$a\hat{i} + 2\hat{j} = 4\hat{i} + b\hat{j}$$

$$a\hat{i} = 4\hat{i} \quad \& \quad 2\hat{j} = b\hat{j}$$

$$\boxed{a=4} \quad \& \quad \boxed{b=2}$$

Example 4 :-

Let  $\vec{a} = 2\hat{i} + 4\hat{j}$ ,  $\vec{b} = 4\hat{i} + 2\hat{j}$ , is  $|\vec{a}| = |\vec{b}|$ ? Are vectors

$\vec{a}$  and  $\vec{b}$  equal?

Sol<sup>n</sup> Given,

$$\vec{a} = 2\hat{i} + 4\hat{j}$$

$$\vec{b} = 4\hat{i} + 2\hat{j}$$

$$|\vec{a}| = \sqrt{2^2 + 4^2}$$

$$= \sqrt{4 + 16}$$

$$= \sqrt{20}$$

$$= 2\sqrt{5}$$

$$|\vec{b}| = \sqrt{4^2 + 2^2}$$

$$= \sqrt{16 + 4}$$

$$= \sqrt{20}$$

$$= 2\sqrt{5}$$



so,  $|\vec{a}| = |\vec{b}|$

But the two vectors are not equal since their corresponding components are distinct.

Example 5 :-

Find the unit vector in the direction of the sum of the vectors  $\vec{a} = \hat{i} + 2\hat{j}$

$$\& \vec{b} = 4\hat{i} + 5\hat{j}$$

Sol<sup>n</sup> Given,

$$\vec{a} = \hat{i} + 2\hat{j}$$

$$\vec{b} = 4\hat{i} + 5\hat{j}$$

The sum of given vectors  $\vec{a}$  and  $\vec{b}$  is,

$$\vec{c} = \vec{a} + \vec{b}$$

$$= (\hat{i} + 2\hat{j}) + (4\hat{i} + 5\hat{j})$$

$$= \hat{i} + 4\hat{i} + 2\hat{j} + 5\hat{j}$$

$$= 5\hat{i} + 7\hat{j}$$

$$|\vec{c}| = \sqrt{(5)^2 + (7)^2}$$

$$= \sqrt{25 + 49}$$

$$= \sqrt{74}$$

Unit vector is

$$\hat{c} = \frac{\vec{c}}{|\vec{c}|}$$

$$= \frac{5\hat{i} + 7\hat{j}}{\sqrt{74}}$$

$$\Rightarrow \hat{c} = \frac{5}{\sqrt{74}} \hat{i} + \frac{7}{\sqrt{74}} \hat{j}$$



Example 6

Subtract the vector  $\vec{b} = 4\hat{i} + 9\hat{j}$  from  $\vec{a} = 3\hat{i} + 20\hat{j}$ .

Sol<sup>n</sup> Given.

$$\vec{a} = 3\hat{i} + 20\hat{j}$$

$$\vec{b} = 4\hat{i} + 9\hat{j}$$

Then,

$$\vec{a} - \vec{b} = (3\hat{i} + 20\hat{j}) - (4\hat{i} + 9\hat{j})$$

$$= 3\hat{i} + 20\hat{j} - 4\hat{i} - 9\hat{j}$$

$$= 3\hat{i} - 4\hat{i} + 20\hat{j} - 9\hat{j}$$

$$= -\hat{i} + 11\hat{j}$$

### Types of Vectors

(1) Zero/null vector : Magnitude is zero initial and terminal points coincide (e.g.,  $\vec{AA}$ ). Denoted by  $\vec{0}$ .  
Direction is indeterminate.

(2) Unit Vector : A vector with unit magnitude (magnitude = 1) used to specify direction.

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

(3) Co-initial vector : Vectors having the same initial point.

(4) Collinear (parallel) Vectors : Vectors parallel to the same line, irrespective of their magnitude or direction.

• Condition :  $\vec{b} = \lambda \vec{a}$  for some scalar  $\lambda$ .

• Component condition :  $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$



- (5) Equal Vectors : Two vectors have the same magnitude and same direction.
- (6) Negative of a Vector : A vector with the same magnitude as  $\vec{a}$  but opposite direction. Denoted as  $-\vec{a}$ .

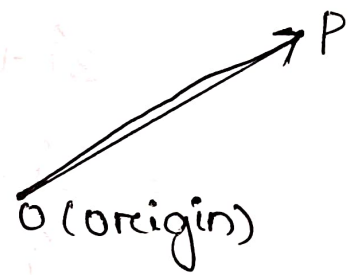
$$\boxed{\vec{BA} = -\vec{AB}}$$

Note :-

position vector of a point P is given as  $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

where,  $\hat{i}$  = Unit vector along x-axis  
 $\hat{j}$  = Unit vector along y-axis  
 $\hat{k}$  = Unit vector along z-axis

$$\therefore |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$



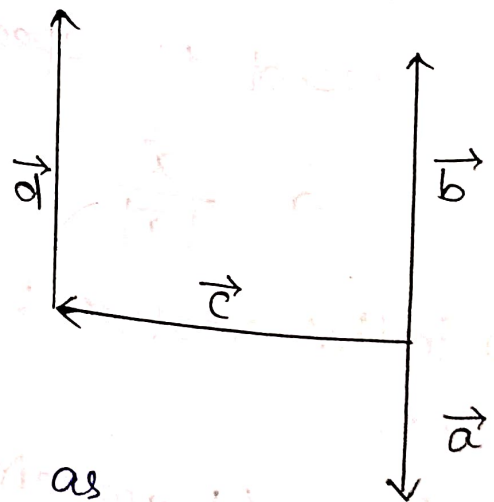
Example 7 :-

Which vectors are

- (i) collinear (ii) collinear but not equal  
 (iii) Equal (iv) co-initial

Sol<sup>n</sup>

- (i)  $\vec{a}, \vec{b}, \vec{c}$  are collinear vectors.  
 (ii)  $\vec{a}$  and  $\vec{c}$  are collinear but not as their directions are not same.  
 (iii)  $\vec{b}, \vec{c}$  are equal vectors.  
 (iv)  $\vec{a}, \vec{b}, \vec{c}$  are co-initial vectors.





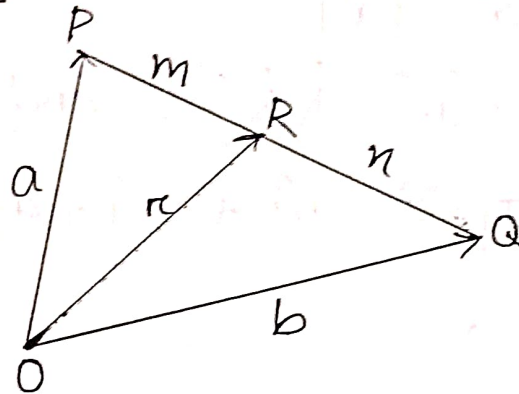
## vector joining Two points :-

If  $P_1(x_1, y_1, z_1)$  and  $P_2(x_2, y_2, z_2)$  are two points :

$$\vec{P_1P_2} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$|\vec{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

## Section formula :-



Position vector  $\vec{r}$  of point R dividing the line segment joining  $P(\vec{a})$  and  $Q(\vec{b})$  in the ratio  $m:n$ ,

- Internal Division :  $\vec{r} = \frac{m\vec{b} + n\vec{a}}{m+n}$

- External Division :  $\vec{r} = \frac{m\vec{b} - n\vec{a}}{m-n}$

- Midpoint formula :  $\vec{r} = \frac{\vec{a} + \vec{b}}{2}$

## Product of two vectors :-

1) Dot product or scalar product

The scalar product of two non-zero vectors  $\vec{a}$  and  $\vec{b}$  with angle  $\theta$  between them is,

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$$

where,  $0 \leq \theta \leq \pi$



Properties :-

1. Commutative :  $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

2. Distributive :  $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

3. Orthogonality (perpendicularity) :

$$\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b} \text{ (for non-zero vectors).}$$

4. Self-product :  $\vec{a} \cdot \vec{a} = |\vec{a}|^2$

5. Unit vectors :  $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$  and  $\hat{i} \cdot \hat{j} = 0$ , etc.

6. Component form : If  $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$  and

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k},$$

then,  $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

7. Angle :  $\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$

Example 8 :-

Evaluate ' $\theta$ ' between the vectors,

$$\vec{a} = \hat{i} + \hat{j} - 2\hat{k}, \vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$$

Sol<sup>n</sup> Given,

$$\vec{a} = \hat{i} + \hat{j} - 2\hat{k}$$

$$\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$$

The angle  $\theta$  between two vectors  $\vec{a} \neq \vec{b}$  is given by

$$\boxed{\cos \theta = \frac{\vec{a} \cdot \vec{b}}{ab}}$$

———— (1)



$$\begin{aligned}
 \text{Now, } \vec{a} \cdot \vec{b} &= (1 \cdot 1) + (1 \cdot 2) + \{(-2) \cdot 3\} \\
 &= 1 + 2 + (-6) \\
 &= 3 - 6 \\
 &= -3
 \end{aligned}$$

$$\begin{aligned}
 a &= |\vec{a}| \\
 &= \sqrt{1^2 + 1^2 + (-2)^2} \\
 &= \sqrt{1+1+4} \\
 &= \sqrt{6}
 \end{aligned}$$

$$\begin{aligned}
 b &= |\vec{b}| \\
 &= \sqrt{1^2 + 2^2 + 3^2} \\
 &= \sqrt{1+4+9} \\
 &= \sqrt{14}
 \end{aligned}$$

Putting  $a$  and  $b$  values in eq<sup>n</sup> (i)

$$\cos \theta = \frac{-3}{\sqrt{6} \sqrt{14}}$$

$$\Rightarrow \theta = \cos^{-1} \left( \frac{-3}{\sqrt{6} \sqrt{14}} \right)$$

$$\Rightarrow \theta = \cos^{-1} \left( \frac{-3}{\sqrt{2 \times 3 \times 2 \times 7}} \right)$$

$$\Rightarrow \theta = \cos^{-1} \left( \frac{-3}{2 \sqrt{21}} \right)$$

Example 9 :-

If  $\vec{a} = \hat{i} - \hat{j} + 5\hat{k}$  and  $\vec{b} = \hat{i} + \hat{j} + \lambda\hat{k}$ , then find  $\lambda$  such that  $\vec{a} + \vec{b}$  and  $\vec{a} - \vec{b}$  are orthogonal.

Sol<sup>n</sup> Given,

$$\vec{a} = \hat{i} - \hat{j} + 5\hat{k}$$

$$\vec{b} = \hat{i} + \hat{j} + \lambda\hat{k}$$



$$1 \quad \vec{a} + \vec{b} = (\hat{i} - \hat{j} + 5\hat{k}) + (\hat{i} + \hat{j} + \lambda\hat{k})$$

$$= 2\hat{i} + (5+\lambda)\hat{k}$$

$$2 \quad \vec{a} - \vec{b} = (\hat{i} - \hat{j} + 5\hat{k}) - (\hat{i} + \hat{j} + \lambda\hat{k})$$

$$= \hat{i} - \hat{j} + 5\hat{k} - \hat{i} - \hat{j} - \lambda\hat{k}$$

$$= -2\hat{j} + (5-\lambda)\hat{k}$$

Now,  $(\vec{a} + \vec{b})$  and  $(\vec{a} - \vec{b})$  are orthogonal ( $90^\circ$ )

$$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 0$$

$$\Rightarrow (2 \cdot 0) + (0 \cdot (-2)) + (5+\lambda)(5-\lambda) = 0$$

$$\Rightarrow 0 + 0 + (5+\lambda) \cdot (5-\lambda) = 0$$

$$\Rightarrow 5^2 - \lambda^2 = 0$$

$$\Rightarrow \lambda^2 = 5^2$$

$$E \quad \Rightarrow \lambda = \pm \sqrt{5^2}$$

$$\Rightarrow \boxed{\lambda = \pm 5}$$

### Projection

• scalar projection of  $\vec{a}$  on  $\vec{b}$  : The length of the shadow of  $\vec{a}$  on  $\vec{b}$ .

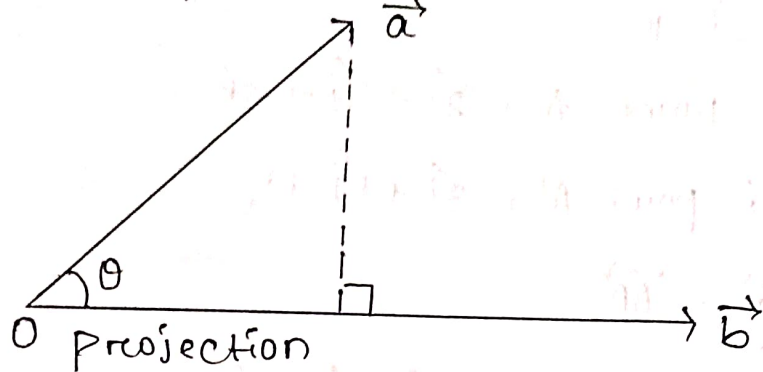
$$\text{Projection} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \vec{a} \cdot \hat{b}$$

• Vector projection of  $\vec{a}$  on  $\vec{b}$  :

$$\text{Vector projection} = \left( \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \right) \vec{b}$$



specific cases for projection :



- If  $\theta = 0$  : projection vector of  $\vec{AB}$  is  $\vec{AB}$  itself.
- If  $\theta = \pi$  : projection vector is  $\vec{BA}$ .
- If  $\theta = \pi/2$  or  $3\pi/2$  : projection vector is  $\vec{0}$ .

Example 10 :-

Find the work done by the force  $\vec{F} = \hat{i} + 2\hat{j} + \hat{k}$  in the direction of displacement  $\vec{a} = 3\hat{i} + 4\hat{j} - \hat{k}$

Sol<sup>n</sup> Given,

$$\vec{F} = \hat{i} + 2\hat{j} + \hat{k}$$

$$\vec{a} = 3\hat{i} + 4\hat{j} - \hat{k}$$

Work done,  $W = \vec{F} \cdot \vec{a}$

$$= (\hat{i} + 2\hat{j} + \hat{k}) \cdot (3\hat{i} + 4\hat{j} - \hat{k})$$

$$= (3 \cdot 1) + (4 \cdot 2) + (-1 \cdot 1)$$

$$= 3 + 8 - 1$$

$$\Rightarrow W = 10 \text{ units.}$$

Example 11 :-

A force  $\vec{F} = 2\hat{i} + 5\hat{j} - \hat{k}$  acts at a point A. The point of application of force moves from the point A to point A'. Where  $2\hat{i} + 4\hat{j} + 5\hat{k}$  and  $3\hat{i} + 5\hat{j} + \hat{k}$  are the position vectors of A and A' respectively. Find the work done.



Sol<sup>n</sup>

Given,

$$\vec{F} = 2\hat{i} + 5\hat{j} - \hat{k}$$

position vector of point A =  $2\hat{i} + 4\hat{j} + 5\hat{k}$

position vector of point A' =  $3\hat{i} + 5\hat{j} + \hat{k}$

$\therefore$  Displacement  $\vec{a} = \vec{AA'}$

$$(3\hat{i} + 5\hat{j} + \hat{k}) - (2\hat{i} + 4\hat{j} + 5\hat{k}) = \hat{i} + \hat{j} - 4\hat{k}$$

work done  $W = \vec{F} \cdot \vec{a}$

$$= (2\hat{i} + 5\hat{j} - \hat{k}) \cdot (\hat{i} + \hat{j} - 4\hat{k})$$

$$= 2 + 5 + 4$$

$$\Rightarrow W = 11 \text{ units}$$

Vector Identities and formulas :-

Based on the dot and cross products, the following identities hold :

$$1. |\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta$$

$$2. |\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$$

$$3. |\vec{a} + \vec{b}|^2 + |\vec{a} - \vec{b}|^2 = 2(|\vec{a}|^2 + |\vec{b}|^2)$$

$$4. (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$$

Triangle projection formula : For a triangle with sides a, b, c and angles A, B, C :

$$\bullet a = b \cos C + c \cos B$$

$$\bullet b = c \cos A + a \cos C$$

$$\bullet c = a \cos B + b \cos A$$



## Vector or cross product :-

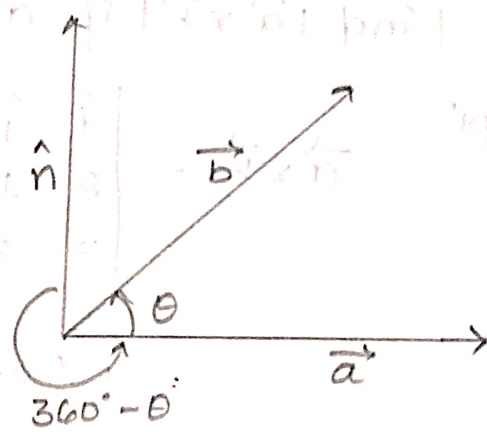
The vector product of two non-zero vectors  $\vec{a}$  and  $\vec{b}$  is a vector  $\vec{c}$  perpendicular to both.

$$\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \sin \theta \hat{n}$$

where,  $\hat{n}$  is a unit vector perpendicular to the plane containing  $\vec{a}$  and  $\vec{b}$ .

Unit normal vector :-

$$\hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$



Properties :-

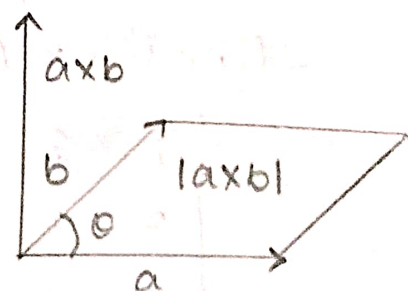
1. Not commutative :  $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$
2. Parallelism :  $\vec{a} \times \vec{b} = \vec{0} \Leftrightarrow \vec{a} \parallel \vec{b}$  (for non-zero vectors)
3. Unit vectors :  $\hat{i} \times \hat{i} = \vec{0}$ ,  $\hat{i} \times \hat{j} = \hat{k}$ ,  $\hat{j} \times \hat{i} = -\hat{k}$  (cyclic order)
4. Determinant form :

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

5. Lagrange's Identity :  $|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$

Geometric Applications :-

- Area of triangle :  $\text{Area} = \frac{1}{2} |\vec{a} \times \vec{b}|$
- Area of parallelogram :  $\text{Area} = |\vec{a} \times \vec{b}|$





Example - 13 :-

Find  $|\vec{a} \times \vec{b}|$  if  $\vec{a} = 2\hat{i} + \hat{j} + 4\hat{k}$ ,  $\vec{b} = 3\hat{i} + 3\hat{j} + 8\hat{k}$

sol<sup>n</sup>

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 4 \\ 3 & 3 & 8 \end{vmatrix}$$

$$= \hat{i}(8-12) - \hat{j}(16-12) + \hat{k}(6-3)$$

$$= -4\hat{i} - 4\hat{j} + 3\hat{k}$$

$$\therefore |\vec{a} \times \vec{b}| = \sqrt{(-4)^2 + (-4)^2 + 3^2}$$

$$= \sqrt{16 + 16 + 9}$$

$$= \sqrt{41}$$

Example - 14 :-

Find a unit vector perpendicular to each of the vectors  $(\vec{a} + \vec{b})$  and  $(\vec{a} - \vec{b})$  where  $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ ,  $\vec{b} = 2\hat{i} + 3\hat{j}$ .

sol<sup>n</sup>

Given,

$$\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$$

$$\vec{b} = 2\hat{i} + 3\hat{j} + \hat{k}$$

$$\text{so, } \vec{a} + \vec{b} = 3\hat{i} + 4\hat{j} + 3\hat{k}$$

$$\vec{a} - \vec{b} = -\hat{i} - 2\hat{j} + \hat{k}$$

Vector  $\vec{c}$  which is perpendicular to  $\vec{a} + \vec{b}$  and  $\vec{a} - \vec{b}$  is

$$= (\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 3 \\ -1 & -2 & 1 \end{vmatrix}$$



$$\Rightarrow \vec{c} = \hat{i}(4+6) - \hat{j}(3+3) + \hat{k}(-6+4)$$

$$\Rightarrow \vec{c} = 10\hat{i} - 6\hat{j} - 2\hat{k}$$

$$\therefore |\vec{c}| = \sqrt{100+36+4}$$

$$= \sqrt{140}$$

$$= 2\sqrt{35}$$

$\therefore$  Required unit vector is

$$\frac{\vec{c}}{|\vec{c}|} = \frac{5\hat{i}}{\sqrt{35}} - \frac{3\hat{j}}{\sqrt{35}} - \frac{\hat{k}}{\sqrt{35}} \quad (\text{Ans.})$$

Scalar Triple product (Box product) :-

The scalar triple product of three vectors  $\vec{a}, \vec{b}, \vec{c}$  is defined as :

$$[\vec{a} \vec{b} \vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c})$$

properties :-

$$1. \text{ Cyclic order : } [\vec{a} \vec{b} \vec{c}] = [\vec{b} \vec{c} \vec{a}] = [\vec{c} \vec{a} \vec{b}]$$

2. Coplanarity : Three vectors are coplanar if and only if  $[\vec{a} \vec{b} \vec{c}] = 0$ .

3. Determinant calculation :

$$[\vec{a} \vec{b} \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

4. Volume : The magnitude of  $[\vec{a} \vec{b} \vec{c}]$  represents the volume of the parallelepiped formed by the three vectors as adjacent edges.



# DIFFERENTIAL EQUATION

→ An equation containing independent variable, dependent variable and differential coefficients of dependent variables with respect to independent variable is called differential equation.

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$$

⤴ Differential coefficient

$y \rightarrow$  Dependent variable

$x \rightarrow$  Independent variable

$$\boxed{\frac{d^2y}{dx^2} + \frac{y}{x} = 0} \rightarrow \text{D.E.}$$

→ If all the differential coefficient are present with respect to independent variable then the differential eqn is called ordinary differential equation. If they are present more than one independent variables then it is called partial differential equation.

$$\text{eg. } \frac{dy}{dx} + \frac{d^2y}{dx^2} + \frac{y^2}{x} = 0$$

⤴ Independent variable

$$\underbrace{\frac{d^2y}{dx^2} + \frac{dy}{dz}}_{\text{out of syllabus}} + x = 0 \rightarrow \text{Independent Variable} = x \neq z$$

→ partial D.E.



→ Differential equation represent a family of curves which share some common characteristics.

e.g.  $\Rightarrow y = mx \rightarrow$  family of straight lines passes through origin

$$\frac{dy}{dx} = \frac{y}{x}$$

$$\Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\Rightarrow \log y = \log x + \log c$$

$$\Rightarrow \log y = \log xc$$

$$\boxed{y = xc}$$

$$y = x$$

$$y = 2x$$

$$y = 5x$$

$$y = 8x$$

$$y = 12x$$

} straight lines passing through origin.

Order and Degree :-

→ The order of highest order differential coefficient present in the equation is called order of differential equation.

$$\frac{d^2y}{dx^2} + \frac{d^3y}{dx^3} + \left(\frac{d^4y}{dx^4}\right)^3 + \frac{d^5y}{dx^5} = 0$$

$$\boxed{\text{Order} = 5}$$



→ The exponent of highest order differential coefficient present in the equation is called degree of differential equation provided if all the differential coefficients are present in polynomial form. Order of a differential equation is always defined while degree need not to be defined.

e.g.  $\left(\frac{d^3y}{dx^3}\right) + \left(\frac{d^2y}{dx^2}\right)^7 + \left(\frac{dy}{dx}\right)^5 = 0$

Degree = 1

Order = 3

e.g.  $\left(\frac{d^5y}{dx^5}\right)^7 + \left(\frac{d^2y}{dx^2}\right)^{100} = 0$

Degree = 7

Order = 5

POLYNOMIAL FORM :-

→ power of differential coefficients is whole no.

e.g.  $\left(\frac{d^3y}{dx^3}\right)' + \left(\frac{d^2y}{dx^2}\right)^8 = \frac{x}{y}$

(in polynomial form)

e.g.  $x^7 y^8 + \left(\frac{d^3y}{dx^3}\right)^8 + \sqrt{\frac{dy}{dx}} = 0$

(Not in polynomial form)

e.g.  $\frac{d^5y}{dx^5} + \left(\frac{d^2y}{dx^2}\right)^{2/3} + y = 0$

(Not in polynomial form)



$$\text{e.g. } \left(\frac{d^3 y}{dx^3}\right)' + \left(\frac{d^2 y}{dx^2}\right)^7 + x^{3/2} = 0$$

(in polynomial form)

$$\text{e.g. } \sin\left(\frac{dy}{dx}\right) = x^5$$

(Not in polynomial form)

$$\text{e.g. } \frac{d^2 y}{dx^2} = \sin(x^5)$$

(in polynomial form)

$$\text{e.g. } \tan^{-1}\left(\frac{d^2 y}{dx^2}\right) = \frac{dy}{dx}$$

(Not in polynomial form)

---

$$\text{e.g. } \sin\left(\frac{dy}{dx}\right) = x^5$$

$$\Rightarrow \frac{dy}{dx} = \sin^{-1}(x^5)$$

order = 1

degree = 1

$$\text{e.g. } \sin\left(\frac{dy}{dx}\right) = \frac{d^2 y}{dx^2}$$

$$\frac{dy}{dx} = \sin^{-1}\left[\frac{d^2 y}{dx^2}\right]$$

order = 2

Degree : not defined

$$\cos\left(x + \frac{dy}{dx}\right) = x^3$$

$$x + \frac{dy}{dx} = \cos^{-1}(x^3)$$

degree = 1

order = 1



$$\text{e.g. } \left(\frac{dy}{dx}\right)^{3/2} = \left(\frac{d^2y}{dx^2}\right)^4$$

Squaring both side

$$\left[\left(\frac{dy}{dx}\right)^{3/2}\right]^2 = \left[\left(\frac{d^2y}{dx^2}\right)^4\right]^2$$

$$\left(\frac{dy}{dx}\right)^3 = \left(\frac{d^2y}{dx^2}\right)^8$$

Order = 2

Degree = 8

$$\text{e.g. } y'' = (y')^{1/3}$$

Cube both side

$$(y'')^3 = ((y')^{1/3})^3$$

$$(y'')^3 = (y')^1$$

Degree = 3

$$\text{e.g. } (y')^{2/3} = (y'')^{7/2}$$

$$\left((y')^{2/3}\right)^6 = \left((y'')^{7/2}\right)^6$$

$$(y')^4 = (y'')^{21}$$

Order = 2

Degree = 1



## General and particular solutions of a differential equation $\rightarrow$

$\rightarrow$  The family of curves obtained by the integrating the given differential equation, free from differential coefficients and containing the arbitrary constants are called general solutions.

$$\frac{dy}{dx} = \frac{y}{x} \rightarrow \text{D.E}$$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\boxed{\log|y| = \log|x| + C}$$

General Solution

Arbitrary constant

## General and particular solutions of a differential equation $\rightarrow$

$\rightarrow$  If the arbitrary constants present in general solution are substituted with some specific values then the obtained equation is called the particular solution.

$$\log y = \log x + C$$

$\downarrow$   
Curve passes through (2,1)

General Solution

$$x=2 \text{ \& } y=1$$

$$-\log 1 = \log 2 + C$$

$$0 = \log 2 + C$$

$$\boxed{C = -\log 2}$$



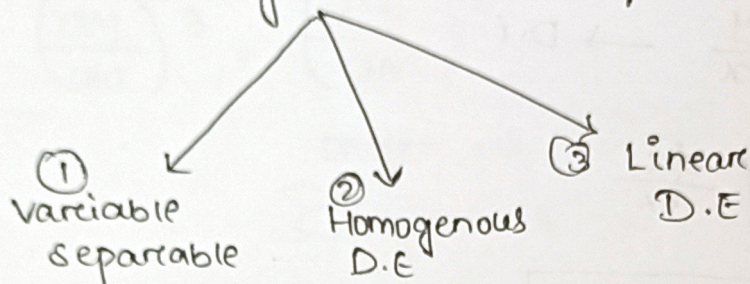
$$\log|y| = \log|x| + C$$

$$\log|y| = \log|x| - \log 2$$

→ Particular solution

\* NO. of arbitrary constants in particular solutions = 0

Methods of solving differential equation



Variable separable :-

The variable separable method is a method used to solve some types of differential equations where variables can be separated on opposite sides.

→ A differential equation is called separable if it can be written as,

$$\frac{dy}{dx} = f(x) g(y)$$

Ex :- solve  $(1+x^2) dy = (1+y^2) dx$ .

Sol Given,

$$(1+x^2) dy = (1+y^2) dx$$

$$\Rightarrow \frac{dy}{1+y^2} = \frac{dx}{1+x^2}$$

~~Now~~



Now integrating both sides,

we get

$$\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$$

$$\Rightarrow \tan^{-1}y = \tan^{-1}x + c \quad (\text{Ans})$$

Homogenous D.E : —

A homogenous d.e is a type of differential eqn where all the terms involve the dependent variable & its derivatives in a way that makes the eqn "scale uniformly."

(i) First-order homogenous D.E :

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

(ii) standard method of solving :

$y = vx$  (where  $v$  is a function of  $x$ )

then,

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

Ex : - Solve  $\frac{dy}{dx} = \frac{x+y}{x}$

Soln

Rewrite,

$$\frac{dy}{dx} = 1 + \frac{y}{x}$$

Let  $y = vx$

$$v + x \frac{dv}{dx} = 1 + v$$



$$x \frac{dv}{dx} = 1$$

separating variables ,

$$\frac{dv}{dx} = \frac{1}{x}$$

integrating ,

$$\int \frac{dv}{dx} = \int \frac{1}{x}$$

$$\Rightarrow v = \ln|x| + C$$

Substitute back  $v = \frac{y}{x}$  :

$$\frac{y}{x} = \ln|x| + C$$

$$\Rightarrow \boxed{y = x(\ln|x| + C)}$$

Linear D.E : -

A linear d.e. is a differential eq<sup>n</sup> in which the dependent variable and its derivatives appear only in the first Power and are not multiplied together.

(1) standard form of first-order linear D.E

$$\frac{dy}{dx} + Pxy = Q$$

where , P & Q are functions of x or constants.

(#) Ex :- ~~solve  $\frac{dy}{dx} + y = e^x$~~

Sol<sup>n</sup> Here,  ~~$P = 1, Q = e^x$~~



## (ii) Method of solving

Step-1: Find the integrating factor (I.F.)

$$\text{I.F.} = e^{\int p dx}$$

Step-2: Multiply the whole eqn by I.F.

Then the left side becomes,

$$\frac{d}{dx} (y \times \text{I.F.})$$

Step-3: Integrate

$$y \times \text{I.F.} = \int (Q \times \text{I.F.}) dx$$

Ex: solve  $\frac{dy}{dx} + y = e^x$

Soln Here,  $p=1$ ,  $Q=e^x$

Integrating factor

$$\text{I.F.} = e^{\int 1 dx} = e^x$$

Multiply the eqn by  $e^x$

$$e^x \frac{dy}{dx} + e^x y = e^{2x}$$

Left side becomes:

$$\frac{d}{dx} (ye^x) = e^{2x}$$

Integrating both sides,

$$ye^x = \int e^{2x} dx$$

$$\Rightarrow ye^x = \frac{e^{2x}}{2} + C$$

$$\therefore y = \frac{e^x}{2} + Ce^{-x}$$



## Examples of Differential eq<sup>n</sup>s : —

Ex-1 : verify that the function  $y = A \sin x + B \cos x$  is a general sol<sup>n</sup> of the D.E  $\frac{d^2 y}{dx^2} + y = 0$

Sol<sup>n</sup> Given,  $\frac{d^2 y}{dx^2} + y = 0$  — (1)

$y = A \sin x + B \cos x$  will be a general sol<sup>n</sup> of (1) if its substitution would give L.H.S = R.H.S.

Consider, differentiating  $y = A \sin x + B \cos x$  — (2)

we get,

$$\frac{dy}{dx} = A \cos x - B \sin x$$

$$\Rightarrow \frac{d^2 y}{dx^2} = -A \sin x - B \cos x \quad \text{--- (3)}$$

Substituting values of  $y$  &  $\frac{d^2 y}{dx^2}$  from (2) & (3) to (1)

we get,

$$\text{L.H.S} = -A \sin x - B \cos x + A \sin x + B \cos x$$

$$= 0 = \text{R.H.S} \quad (\text{proved})$$

Ex-2 : Evaluate  $\frac{dy}{dx} = e^{x-2y} + x^4 e^{-2y}$

Sol<sup>n</sup> Given D.E,

$$\frac{dy}{dx} = e^{x-2y} + x^4 e^{-2y}$$

$$= (e^x + x^4) e^{-2y}$$

Now separating the variable, we get

$$(e^x + x^4) dx = e^{2y} dy$$



Integrating both sides, we get

$$\int (e^x + x^4) dx = \int e^{2y} dy$$

$$\Rightarrow e^x + \frac{x^5}{5} = \frac{e^{2y}}{2} + C \quad (\text{Ans})$$

Ex 3 :  $\log\left(\frac{dy}{dx}\right) = a_1x + a_2y$

Sol<sup>n</sup> Given D.E.,

$$\log\left(\frac{dy}{dx}\right) = a_1x + a_2y$$

$$\Rightarrow \frac{dy}{dx} = e^{a_1x + a_2y}$$

$$\Rightarrow \frac{dy}{e^{a_2y}} = e^{a_1x} dx \quad (\text{separating the variables})$$

Integrating both sides,

we get  $\int \frac{dy}{e^{a_2y}} = \int e^{a_1x} dx$

$$\Rightarrow \frac{e^{-a_2y}}{-a_2} = \frac{e^{a_1x}}{a_1} + C$$

$$\Rightarrow \frac{e^{a_1x}}{a_1} + \frac{e^{-a_2y}}{a_2} + C = 0 \quad (\text{Ans}).$$

Ex 4 : solve  $5x dy - 2y dx = 2x^2 dy$

Sol<sup>n</sup> Given D.E.,

$$5x dy - 2y dx = 2x^2 dy$$

Separating the variables, we get

$$(5x - 2x^2) dy = 2y dx$$

$$\Rightarrow \frac{dy}{2y} = \frac{dx}{x(5-2x)}$$



$$\Rightarrow \frac{dy}{2y} = \frac{1}{5} \left[ \frac{1}{x} + \frac{2}{5-2x} \right] dx$$

Integrating both sides, we get

$$\int \frac{dy}{2y} = \frac{1}{5} \int \left[ \frac{1}{x} + \frac{2}{5-2x} \right] dx + \log c$$

$$\Rightarrow \frac{1}{2} \log y = \frac{1}{5} \left[ \log x + \frac{2}{(-2)} \log(5-2x) \right] + \log c$$

$$\Rightarrow \log \sqrt{y} = \frac{1}{5} \left[ \log \left( \frac{x}{5-2x} \right) \right] + \log c$$

$$\Rightarrow \sqrt{y} = c \left( \frac{x}{5-2x} \right)^{1/5} \quad (\text{Ans})$$

— x —